Meson Theory of Hyperdeuterons

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1 Problem

Estimate the relative binding energies of the 64 possible pairs of baryons in the basic octet: \( n, p, \Lambda, \Sigma^-, \Sigma^0, \Sigma^+, \Xi^-, \Xi^0 \).

For this, use a simplified one-pion-exchange model that the nuclear force is entirely due to exchanges of a single \( \pi \) meson, and that the operator \( g^2 \tau_1 \cdot \tau_2 \) characterizes the charge independence of this interaction.\(^1\) Here \( g \) is a coupling constant, and \( \tau \) is the isospin-1 operator (because pions form an \( I = 1 \) multiplet). That is, ignore electromagnetic effects and spin-dependent effects.

(A harder version of the problem would be to deduce that \( g^2 \tau_1 \cdot \tau_2 \) is the appropriate operator.)

A hint is that the Hamiltonian relevant to binding of the dibaryons is \( H \propto g^2 \tau_1 \cdot \tau_2 \). Hence, we should consider the matrix elements \( \langle B_1B_2 | \tau_1 \cdot \tau_2 | B_1B_2 \rangle \), where \( B \) is any member of the baryon octet. As for electricity, we infer that a negative matrix element implies an attractive force, and bound states, while a positive matrix element implies repulsion.

Recall that
\[
\tau_1 \cdot \tau_2 = \frac{1}{2} (\tau_{+1,1} \tau_{-2} + \tau_{-1,2} \tau_{+2}) + \tau_{z,1} \tau_{z,2},
\]
where
\[
\tau_+ \equiv \tau_x + i\tau_y, \quad \tau_- \equiv \tau_x - i\tau_y,
\]
and
\[
\tau_+ |t, t_z\rangle = \sqrt{(t + t_z + 1)(t - t_z + 1)} |t, t_z + 1\rangle,
\]
\[
\tau_- |t, t_z\rangle = \sqrt{(t + t_z)(t - t_z + 1)} |t, t_z - 1\rangle.
\]
Thus \( \tau_+ |n\rangle = |p\rangle \), and \( \tau_- |p\rangle = |n\rangle \). Of course, \( \tau_z |p\rangle = \frac{1}{2} |p\rangle \) and \( \tau_z |n\rangle = -\frac{1}{2} |n\rangle \). (That is, the \( \tau \) operators can be represented by Pauli spin matrices times 1/2.)

Show that this model predicts that the isotriplet \( pp \), \( (np+pm)/\sqrt{2} \), \( nn \) is unbound [having equal, positive matrix elements (charge independence)], while the isosinglet \( (np-pm)/\sqrt{2} \) is bound. Show that this state is an eigenstate of \( \tau_1 \cdot \tau_2 \) with eigenvalue \(-\frac{3}{4}\).

Having examined 4 of the 64 dibaryon states, turn to the other 60. Note that
\[
\tau_1 \cdot \tau_2 = \frac{1}{2} \left( \tau^2 - \tau_1^2 - \tau_2^2 \right).
\]

Also, charge independence means you don’t have to look at each of the 64 pairs separately, but you can more simply consider pairs of isospin multiplets, each of which leads to one or more multiplets of total isospin exactly as for combinations of ordinary spin. For this, note

\(^1\)If this interaction is represented by a Feynman diagram with single pion exchange, then each of the \( BB\pi \) vertices has strength \( g\tau \).
that the nucleons, \( N \), and the cascade particles, \( \Xi \), each form an isodoublet, the \( \Lambda \) is an isosinglet, and the \( \Sigma \)'s form an isotriplet.

Give the isospin wavefunctions of the candidate bound states.

I found that 11 of the 64 pairs should have bounds states, and that none of these would be more weakly bound than the deuteron.

No dibaryon bound state other than the deuteron has ever been observed, although searches continue.\(^2\) The lightest known hypernucleus is \( ^3\Lambda H \),\(^3\) and even its antiparticle has been observed.\(^4\) A \( \Sigma \)-hypernucleus is \( ^4\Sigma\Lambda He \).\(^5\) A handful of examples of hyper-He nuclei containing two \( \Lambda \)'s have been reported.\(^6\)
2 Solution

Start with the two-nucleon states: \( pp, \, nn \) and \( np \). The \( pp \) diagram gets a factor \( \frac{1}{2} \) at each vertex for an overall strength of \( \frac{1}{2} \) (times \( g^2 \) which we ignore when comparing strengths). For the \( nn \) diagram we get \((-\frac{1}{2})(-\frac{1}{2}) = \frac{1}{4} \) also. Now for the \( np \) case we get \((-\frac{1}{2})(\frac{1}{2}) = -\frac{1}{4} \) for the diagram with \( \pi^0 \) exchange, and \((\frac{1}{2})(1)(1) = \frac{1}{2} \) for the diagram with \( \pi^+ \) exchange. But these two diagrams interfere, so we add the amplitudes, and the result is again \( \frac{1}{2} \). Charge independence!

We have glossed over an important point in the above. There are really two kinds of \( np \) states: \((|pn\rangle + |np\rangle)/\sqrt{2}\) and \((|pn\rangle - |np\rangle)/\sqrt{2}\). The charge-independent result above is for the first state, which we recognize as the partner of \(|pp\rangle \) and \(|nn\rangle \) in the symmetric isospin-1 triplet. The second \( np \) state is the antisymmetric isospin-0 singlet. Show that this state is an eigenstate of \( \tau_1 \cdot \tau_2 \) with eigenvalue \(-\frac{3}{4}\).

Thus, we infer that the single \( np \) state is bound with a relative strength of \(-3/4 \) units, while the triplet \( pp, \, np, \, nn \) states are unbound (but a continuum level exists at +1/4 relative units).

To deal with all 64 pairs of dibaryons from the basic spin-1/2 octet, \( n, \, p, \, \Lambda, \Sigma^-, \Sigma^0, \Sigma^+, \Xi^-, \Xi^0, \), we need a more compact analysis. For this, we note that for \( \tau = \tau_1 + \tau_2 \),

\[
(\tau_1 + \tau_2)^2 = \tau^2 = \tau_1^2 + \tau_2^2 + 2\tau_1 \cdot \tau_2, \tag{1}
\]

\[
\tau_1 \cdot \tau_2 = \frac{1}{2} \left( \tau^2 - \tau_1^2 - \tau_2^2 \right) = \frac{1}{2} \left[ (\tau(\tau + 1) - (\tau_1(\tau_1 + 1) - (\tau_2(\tau_2 + 1)] \right), \tag{2}
\]

noting that the expectation value of the (iso)spin operator \( \tau^2 \) is \( \tau(\tau + 1) \). Thus, the strength of the \( \tau_1 \cdot \tau_2 \) interaction is the same for all members of a multiplet of total isospin \( \tau \), and in the simplest model is the same for any dibaryon isospin multiplet of the same \( \tau \).

So, we consider the possible dibaryon isospin multiplets.

1. The \( 1/2 \times 1/2 \) multiplets are \( NN, \, \Xi\Xi \) and \( N\Xi \), which lead to \( \tau = 0 \) and \( \tau = 1 \) multiplets with \( \tau_1 = \tau_2 = 1/2 \).

\[
\tau = 0: \quad \tau_1 \cdot \tau_2 = \frac{1}{2} \left[ 0(0+1) - \left( \frac{1}{2} \right) \left( \frac{1}{2} + 1 \right) - \left( \frac{1}{2} \right) \left( \frac{1}{2} + 1 \right) \right] = -\frac{3}{4}. \tag{3}
\]

\[
\tau = 1: \quad \tau_1 \cdot \tau_2 = \frac{1}{2} \left[ 1(1+1) - \left( \frac{1}{2} \right) \left( \frac{1}{2} + 1 \right) - \left( \frac{1}{2} \right) \left( \frac{1}{2} + 1 \right) \right] = \frac{1}{4}. \tag{4}
\]

This model suggests that there would be bound isosinglet states \( (pn-np)/\sqrt{2} \) (deuteron), \( (\Xi^0\Xi^- - \Xi^-\Xi^0)/\sqrt{2} \) and \( (p\Xi^- - n\Xi^0)/\sqrt{2} \).

2. The \( 1/2 \times 0 \) multiplets are the \( N\Lambda \) and \( \Xi\Lambda \) states, with \( \tau = 1/2, \, \tau_1 = 1/2 \) and \( \tau_2 = 0 \).

\[
\tau = \frac{1}{2}: \quad \tau_1 \cdot \tau_2 = \frac{1}{2} \left[ \left( \frac{1}{2} \right) \left( \frac{1}{2} + 1 \right) - \left( \frac{1}{2} \right) \left( \frac{1}{2} + 1 \right) - (0)(0+1) \right] = 0. \tag{5}
\]

These states are not bound in this model.
3. The $0 \times 0$ multiplet is the state $\Lambda\Lambda$, with $\tau = 0 = \tau_1 = \tau_2 = 0$.

$$\tau = 0 : \quad \mathbf{\tau}_1 \cdot \mathbf{\tau}_2 = \frac{1}{2} \left[ (0) (0 + 1) - (0) (0 + 1) - (0) (0 + 1) \right] = 0. \quad (6)$$

This state is not bound in this model.

4. The $1 \times 0$ multiplet is the states $\Sigma\Lambda$, with $\tau = 1 = \tau_1$ and $\tau_2 = 0$.

$$\tau = 1 : \quad \mathbf{\tau}_1 \cdot \mathbf{\tau}_2 = \frac{1}{2} \left[ (1) (0 + 1) - (1) (1 + 1) - (0) (0 + 1) \right] = 0. \quad (7)$$

These states are not bound in this model.

5. The $1/2 \times 1$ multiplets are the $N\Sigma$ and $\Xi\Sigma$ states, with $\tau = 1/2$ or $3/2$, $\tau_1 = 1/2$ and $\tau_2 = 1$.

$$\tau = \frac{1}{2} : \quad \mathbf{\tau}_1 \cdot \mathbf{\tau}_2 = \frac{1}{2} \left[ \left( \frac{1}{2} \right) \left( \frac{1}{2} + 1 \right) - \left( \frac{1}{2} \right) \left( \frac{1}{2} + 1 \right) - 1 (1 + 1) \right] = -1, \quad (8)$$

$$\tau = \frac{3}{2} : \quad \mathbf{\tau}_1 \cdot \mathbf{\tau}_2 = \frac{1}{2} \left[ \left( \frac{3}{2} \right) \left( \frac{3}{2} + 1 \right) - \left( \frac{1}{2} \right) \left( \frac{1}{2} + 1 \right) - 1 (1 + 1) \right] = \frac{1}{2}. \quad (9)$$

6. The $1 \times 1$ multiplets are the $\Sigma\Sigma$ states, with $\tau = 0, 1$ and $\tau_1 = \tau_2 = 1$.

$$\tau = 0 : \quad \mathbf{\tau}_1 \cdot \mathbf{\tau}_2 = \frac{1}{2} \left[ 0 (0 + 1) - 1 (1 + 1) - 1 (1 + 1) \right] = -2, \quad (10)$$

$$\tau = 1 : \quad \mathbf{\tau}_1 \cdot \mathbf{\tau}_2 = \frac{1}{2} \left[ 1 (1 + 1) - 1 (1 + 1) - 1 (1 + 1) \right] = -1, \quad (11)$$

$$\tau = 2 : \quad \mathbf{\tau}_1 \cdot \mathbf{\tau}_2 = \frac{1}{2} \left[ 2 (2 + 1) - 1 (1 + 1) - 1 (1 + 1) \right] = 1. \quad (12)$$

This model suggests that the most tightly bound state would be isosinglet $(\Sigma^+\Sigma^- - \Sigma^0\Sigma^0 + \Sigma^-\Sigma^+)/\sqrt{3}$, and the isotriplet $\Sigma^+\Sigma^0, \Sigma^+\Sigma^- , \Sigma^0\Sigma^-$ is also bound.\(^7\)

Taking Coulomb effects into account (which don’t conserve isospin), the $\Sigma^+\Sigma^-$ part of the isosinglet would be the most tightly bound dibaryon in this model.

Unfortunately, the data do not support this model.

\(^7\)Note that the state $\Sigma^0\Sigma^0$ does not contribute to the $\Sigma\Sigma$ isotriplet. (Similarly, the isovector meson $\rho^0$ does not decay to $\pi^0\pi^0$.)