DETECTING ELEMENTARY PARTICLES

The best reference for the physics of this lecture remains the book of Rossi.

We can only observe directly those elementary particles which interact with a chunk of matter. As the bulk properties of matter are governed by electromagnetism it is not surprising that all particle detectors are based on the relatively low-energy electrical interactions of charged particles with matter. Detectors of neutral particles (protons, neutrons, neutrinos...) rely on a high energy interaction to convert the neutral particle into 2 n charged particles, which are really what is detected.

INTERACTION OF A HIGH ENERGY CHARGED PARTICLE WITH ATOMS

Long range electromagnetic interactions with the electrons and nucleus of an atom affect the high energy particle in two ways of interest:

- Its energy is reduced
- Its motion is deflected.

As for the atom, the energy it absorbs goes almost entirely into the excitation and ionization of the electrons. Most particle detectors essentially observe these excited electrons.

We now give some semi-classical estimates of the excitation process.

\[
\text{The maximum force on the moving charge is } F_{\text{max}} = \frac{2e^2}{l}\]

where \( l \) = impact parameter
\( = \) distance of closest approach

This force is applied for a characteristic time \( \Delta t = \frac{2l}{v} \), time that \( F \geq \frac{F_{\text{max}}}{2} \)

Thus an impulse \( \Delta p = F_{\text{max}} \Delta t = \frac{2e^2}{lv} \) is transmitted between the two particles.

Exercise: In the limit \( l \to 0 \), show that this result is "exact" by an application of Gauss's law.

It is important to note that we get the same result on calculating the impulse on the target particle. The field of a moving charge is stronger according to Einstein, so \( F_{\text{max}} \) on target = \( \gamma \frac{e^2}{l} \).

But the field is Lorentz contracted along the direction of motion, so \( \Delta t \approx \frac{2l}{\gamma v} \Rightarrow \Delta p \) as before!
The moving charge is deflected by angle \( \theta = \frac{Ap}{p} = \frac{2Ze^2}{b\gamma m_0v^2} \)

Exercise: Compare with gravity. Note that we might think of a proton as the limit \( n \rightarrow 0, \gamma \rightarrow \infty \) but \( E = m_0\gamma c^2 \) is constant.

Then \( \theta \rightarrow 0 \) for deflection of photons due to a force derived from a 4-vector potential. What is the result for scalar and tensor potentials? Do you have to know about general relativity to answer this?

The moving particle loses a small amount of energy in the above interaction, which is more readily calculated as the energy gain of the target particle, namely:

\[ AE = \frac{Ap^2}{2M} = \frac{2Ze^4}{M_0b^2v^2} \]

It is interesting to compare \( \theta \) and \( AE \) due to scattering off the electrons or the nucleus of the atom.

\( AE \) electron \( \approx \frac{Ze^4}{M_0} \), while \( AE \) nucleus \( \approx \frac{Ze^4}{M_{\text{nucleus}}} \)

Now \( \frac{Z}{M_{\text{nucleus}}} \ll \frac{1}{M_0} \) so energy is lost primarily to scattering off the electrons. However, in the case of \( \theta \), the scatters off the \( Z \) different directions, and add up to a net angle \( \rightarrow 120^\circ \) on average (a random walk process). So scattering off the nucleus is more important in deflecting the moving charge.

**Energy loss per gram/cm\(^2\)**

We sketch an argument to estimate the total energy lost to atomic electrons by a fast charged particle traversing a block of material. It is customary to measure the thickness of the material not in cm, but in grams/cm\(^2\) = length/density. Thus:

\[ AE_{\text{total}} = \left( \text{# of electrons in thickness } dAx \right) \left( \text{energy loss to an electron} \right) \]

\[ \text{# of } Z's = \frac{N_0}{A} \cdot Z \cdot dAx \]

\( N_0 = \text{Avogadro's number} \)

\( A = \text{atomic number} \)

We found the energy loss to an electron if the impact parameter is \( b \). The probability of this is just \( 2\pi bdb/1\text{cm}^2 \)

Altogether:

\[ \frac{dE}{dx} = \frac{N_0}{A} \left( \int_{b_{\text{min}}}^{b_{\text{max}}} 2\pi bdb \cdot \frac{2Ze^4}{M_0b^2v^2} \right) = \frac{9\pi N_0}{A} \frac{Z e^4}{M_0b_{\text{max}}^2v^2} \frac{\mathcal{L}_n}{(6\text{max})} \]
There are limits on the range of impact parameter $b$ which can contribute to the energy loss.

From Heisenberg we learn $b_{\text{min}} \approx \frac{h}{P_{\text{cm}}} \approx \frac{h}{P}$

$b$ is transverse to $\mathbf{P}$ so this may at first seem an unusual application of the uncertainty rules. But $bP = \text{angular momentum}$, which will have an uncertainty $\approx \frac{h}{P}$ for initial states not prepared with definite $L$, as in shooting a particle into a block.

For a collision with an electron, $\gamma_{\text{cm}} \approx \gamma$, and $P_{\text{cm}} = \gamma m_e \gamma_{\text{cm}} \approx \gamma m_e \gamma$

Hence $b_{\text{min}} \approx \frac{h}{m_e \gamma}$

On the other hand, if $b$ is too large, the impulse is applied so slowly that the electron does not act independently of the rest of the atom. The atom as a whole recoils, absorbing negligible energy (due to its high mass).

That is, we suppose $A_{\text{max}} \approx 2 \frac{b_{\text{max}}}{\gamma}$

The factor $\frac{1}{\gamma}$ occurs because of the Lorentz contraction of the field of the moving charged, noted above.

Now $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ where $v = \text{frequency}$

$I = \langle \gamma v \rangle = \text{average electron binding energy} = \text{ionization potential}$

Thus $b_{\text{max}} \approx \frac{\hbar \gamma v}{2 I}$

And $\frac{dE}{dx} \approx 4\pi \frac{N_e \gamma^4}{A} \frac{Z^2 e^4}{m_e v^2} \left( \frac{\hbar \gamma^2 \mathcal{B} v^2}{I} \right)$

Compare to the Bethe-Block result $4\pi \frac{N_e \gamma^4}{A} \frac{Z^2 e^4}{m_e v^2} \left( \frac{\hbar \gamma^2 \mathcal{B} v^2}{I} - \frac{2v^2}{c^2} \right)$

$dE/dx$ falls like $1/v^2$ if $v < c$, $\approx \gamma$ if $\gamma \sim 1$; it reaches a broad minimum at $\gamma \sim 3-4$; and rises slowly for $\gamma > 4$ - the relativistic rise. A particle with $\gamma \sim 370v$ is called a minimum ionizing particle. $dE/dx \sim 1-2 \text{MeV per } 6\text{MeV/cm}^2$ at minimum. See tables appended to this lecture.
AT VERY SMALL VELOCITIES OUR EXPRESSION DIVERGES. HOWEVER THE WHOLE DERIVATION DOESN'T MAKE MUCH SENSE IF \( v < \frac{1}{\sqrt{3}} \). THE VERY LOW VELOCITY LIMIT HAS BECOME OF INTEREST LATELY BECAUSE OF MAGNETIC MONOPOLE SEARCHES. (SEE DREU ET AL. PHYS. REV. LETT. 50, 644 (1983)).

[EXERCISE: WHAT IS THE \( \frac{dE}{dx} \) LOSS FOR A MAGNETIC MONOPOLE MOVING THROUGH MATTER, IF \( \frac{1}{\sqrt{3}} < v < 1 \)?]

THE FIGURES SHOW CALCULATED AND OBSERVED \( \frac{dE}{dx} \) LOSS CURVES IN ARGON GAS.

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**Fig. 11.** The most probable number of ion pairs/cm for argon and xenon at STP, as a function of \( P(\text{GeV/c}) \) for pions, kaons, and protons. Not shown are the curves for electrons, which would be essentially flat at the asymptotic values governed by the density effect.

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THIS INDICATES THE POSSIBILITY OF DETERMINING THE PARTICLE'S MASS (ASSUMING CHARGE = 2) BY A MEASUREMENT OF MOMENTUM \( P \) AS WELL AS \( \frac{dE}{dx} \). THIS IS RARELY POSSIBLE FOR \( \chi > 4 \) DUE TO THE RELATIVISTIC RISE. BUT FOR \( \chi < 4 \) SUCH MASS IDENTIFICATION IS READILY ACHIEVED, AND IN FACT THE FIRST ESTIMATES OF THE \( \pi \) AND K MESON MASSES CAME FROM \( \frac{dE}{dx} \) MEASUREMENTS OF TRACKS IN NUCLEAR EMULSIONS.

THE RELATIVISTIC RISE OCCURS BOTH BECAUSE \( \beta_{\text{max}} \) INCREASES AND \( \beta_{\text{min}} \) DECREASES AS \( \gamma \rightarrow \infty \). THE FORMER CAN BE ATTRIBUTED TO THE RELATIVISTIC GROWTH OF THE ELECTRIC FIELD OF A MOVING CHARGE. HOWEVER, ONCE \( \beta_{\text{max}} \) BECOMES LARGER THAN THE INTERATOMIC SPACING, DUE ATOM SHIELDS ANOTHER, \( \beta_{\text{min}} \) DOES NOT INCREASE FURTHER— THE DENSITY EFFECT. SO THE RISE IS MUCH MORE PROMINENT IN GASES THAN IN SOLIDS OR LIQUIDS.

PART OF THE RELATIVISTIC RISE IS ASSOCIATED WITH COHERENT RESPONSE OF THE MATERIAL TO THE ELECTRIC FIELD OF THE MOVING CHARGE: THE CERENKOV EFFECT DISCUSSED BELOW.
ROUGHLY ONE HALF THE ENERGY DEPOSITED GOES INTO IGNIZING ELECTRONS. THE REST MERELY EXCITES THE ATOMS TO HIGHER BOUND STATES. THESE MAY DE-EXCITE BY PHOTON EMISSION CAUSING SCINTILLATION LIGHT.

IN MOST MATERIALS ONE ELECTRON IS IONIZED FOR EACH 25-35 eV DEPOSITED. HOWEVER, MANY OF THESE IONIZATIONS ARE NOT DIRECTLY DUE TO THE FIELD OF THE MOULM CHARGE. AN ELECTRON EJECTED BY THE EFFECT OF THE MOULM CHARGE (PRIMARilly IONIZATION) TYPICALLY CARRIES AWAY ENOUGH KINETIC ENERGY TO CAUSE 2-3 ADDITIONAL IONIZATIONS. THE SO-CALLED \( \delta \)-RAYS ARE JUST VERY ENERGETIC PRIMARY IONIZATION ELECTRONS. WE FOUND

\[ E \sim V_b \frac{1}{b} \] FOR THESE ELECTRONS. THE PROBABILITY OF IMPACT PARAMETER \( b \) IS \( \pi b^2 \delta \), WHICH THEN YIELDS \( \Pi(E) \sim \frac{dN}{dE} \frac{1}{E^2} \) AS THE \( \delta \)-RAY ENERGY SPECTRUM. THE FLUCTUATIONS ABOUT THE MEAN ENERGY DEPOSITION DUE TO \( \delta \)-RAYS HAS BEEN STUDIED BY YAVILOV AND LANDAU, AND LEAD TO A NON-GAUSSIAN DISTRIBUTION.

AS USUAL, WITH STATISTICAL PROCESSES THE MORE SAMPLES TAKEN (MORE PRIMARY IONIZATION = THICKER MATERIAL), THE MORE NEARLY GAUSSIAN THE TOTAL ENERGY LOSS DISTRIBUTION BECOMES.

MULTIPLE COULOMB SCATTERING

AS WELL AS LOSING ENERGY WHILE TRAVERSING MATERIAL, A HIGH ENERGY CHARGED PARTICLE SUFFERS MANY SMALL DEVIATIONS, DUE MAINLY TO COULOMB SCATTERING OFF THE NUCLEUS. WE FOUND

\[ d\delta(\Theta) = \frac{8\pi e^2}{b^2 m_0 v^2} \] IT IS AMUSING TO NOTE THAT THIS QUICKLY LEADS TO THE RUTHERFORD SCATTERING FORMULA AT SMALL ANGLES.

\[ d\Sigma = 2\pi b \delta b = 2\pi \cdot 2\pi \cdot \frac{4\pi e^4}{b^2 m_0 v^2} \frac{d\delta}{d\Theta} \propto \frac{d\Sigma}{d\delta} = \frac{2e^4}{8m_0 v^2} \Theta^2 \frac{d\Theta}{d\delta} \frac{d\Sigma}{d\Theta} \frac{4(\pi r)}{(\Theta^2)^2} \]

AN INTERESTING QUANTITY IS THE TOTAL CHANGE IN ANGLE AFTER MANY COULOMB INTERACTIONS = \( \Theta_0 \). AS MANY SCATTERS OCCUR TO THE LEFT AS TO THE RIGHT, UP OR DOWN, SO \( \Theta_0 = 0 \) ON AVERAGE. BUT \( \langle \Theta_0^2 \rangle \) WILL BE NON-VANISHING. IT GROWS WITH THE THICKNESS OF MATERIAL TRAVERSED, AS IN A RANDOM WALK. THAT IS,

\[ \langle \Theta_0^2 \rangle = \delta \Theta^2 \times \] \( \mu \) = THICKNESS IN cm/cm², SAY.

WHERE \( \delta \Theta^2 = \frac{d\Theta(\Theta)}{d\Theta} \) CHANGE IN \( \Theta \) WHILE CROSSING UNIT THICKNESS

\[ \delta \Theta^2 = \int \Theta^2 \rho(\Theta) d\Sigma \approx 2\pi \int \Theta^3 \rho(\Theta) d\Theta \] AS SMALL ANGLES DOMINATE.
\[ P(\Theta) = \text{Proportion of Scattering by } \Theta \text{ in a Single Collision} \quad \text{of Nuclei per } \text{cm}^2 \]
\[
= \frac{N_0}{A} \int_{\text{Rutherford}} \frac{d\sigma}{d\Omega} = \frac{N_0}{A} \left( \frac{Ze^2}{\mu^2} \right)^2 \frac{1}{\Theta^4}
\]

\[ \text{and } \gamma^2 = 8\pi \frac{N_0}{A} \left( \frac{Ze^2}{\mu^2} \right)^2 \int \frac{d\theta}{\Theta} = 8\pi \frac{N_0}{A} \left( \frac{Ze^2}{\mu^2} \right)^2 l_n \left( \frac{\Theta_{\text{max}}}{\Theta_{\text{min}}} \right) \]

As before, there are limits to the plausible range of scattering.

Recall that \( \Theta \approx \frac{1}{\lambda} \) so \( \frac{\Theta_{\text{max}}}{\Theta_{\text{min}}} = \frac{b_{\text{max}}}{b_{\text{min}}} \). We need the relevant limits on the impact parameter for nuclear coulomb scattering.

\( b_{\text{max}} \approx \text{Radius of Atom} \); otherwise the electrons screen the nucleus

\[ \left( \frac{r_a}{2} \right) \approx \text{Bare Radius in the Thomas-Fermi Model} = \frac{r_0}{\alpha} \approx \frac{r_0}{2^{1/3}} \left( \frac{Z_e^2}{M_N c^2} \right) \]

\( b_{\text{min}} \approx \text{Radius of Nucleus} \approx A^{1/3} \cdot \text{Fermi} \)

\[ \frac{b_{\text{max}}}{b_{\text{min}}} = \frac{r_a}{(2A)^{1/3} \alpha^2 \cdot \text{Fermi}} = \frac{2.8(137^2)^{1/3}}{2^{1/3} \alpha^2} \approx \left( \frac{200}{2^{1/3}} \right) \]

\[ \gamma^2 \approx 16\pi \frac{N_0}{A} \frac{Z^2}{(P\beta)} \left( \frac{Ze^2}{\mu^2} \right)^2 l_n \left( \frac{200}{2^{1/3}} \right) \]

This is commonly rewritten as

\[ \gamma^2 = 4\pi \frac{N_0}{A} \frac{Z^2}{\left( \frac{Ze^2}{M_N c^2} \right)^2} \left( \frac{183}{2^{1/3}} \right) \cdot \frac{4\pi}{\alpha^2} \left( \frac{M_N c^2}{P\beta} \right) \left( \frac{1}{P\beta} \right)^2 \]

\[ \approx \frac{1}{\chi_0} \left( \frac{21 \text{ MeV/c}}{c} \right)^2 \]

\[ \chi_0 \equiv \text{Radiation Length}, \text{ a characteristic property of the material.} \]

Then \( \langle \Theta_0^2 \rangle = \left( \frac{21 \text{ MeV/c}}{P\beta} \right)^2 \frac{Z}{\chi_0} \quad \Theta_0 \text{ RMS} = \frac{21}{P\beta} \sqrt{\frac{Z}{\chi_0}} \)

It is often more natural to deal in the projected scattering angle onto the x-z and y-z planes.

\[ \langle \Theta_x^2 \rangle + \langle \Theta_y^2 \rangle = \langle \Theta_0^2 \rangle \quad \text{so} \quad \Theta_x = \Theta_y = \frac{15}{P\beta} \sqrt{\frac{Z}{\chi_0}} \]

For \( P \) measured in MeV/c.
As the particle's angle changes due to multiple Coulomb scattering, a net transverse displacement is generated, also by a random walk process. Can you show that

\[ \Delta x = \Delta y = \sqrt{\langle y^2 \rangle} = \frac{1}{\sqrt{3}} \theta \Delta z \]

by statistical consideration?

\[ \Delta y \] and \[ \theta \] etc. obey approximate Gaussian distributions, with standard deviations being the quantities deduced above. However, there are long non-Gaussian tails due to occasional very hard nuclear scatterings not well described by taking \[ \theta \] as above.

We now consider various detectors in common use.

**Scintillation Detectors**

We indicated above that about \( \frac{1}{2} \) the energy deposited in a block of material by a passing charged particle goes into excitation of the atoms to higher electronic levels. Most excited atoms have reasonable probability to de-excite by emission of a photon (rather than, say, transferring energy to the vibration of the material as a whole). This light output is called scintillation. An important class of detectors collects this light as the signal due to the passage of an elementary charged particle.

The conversion of the collected light into an electrical signal is done in a photomultiplier tube, a kind of vacuum tube diode. Light enters through a glass window and strikes a material such as antimony-cesium, which has a strong photoelectric effect (70% probability of ejection of one electron into the vacuum tube). The photo-electrons are pulled by an electric field onto a dynode (also of Sb-Cs, or Br-Cs), where secondary emission occurs. Typically 3-4 secondary electrons are emitted for each electron incident on a dynode. A tube with 10 stages achieves a gain of 10^5.

![Fig. 4.2. Scintillation counter. A particle passing through the scintillator produces light which is transmitted through a light pipe onto a photomultiplier.](image-url)
So for an initial signal of only 10 photoelectrons, there are 10^6 electrons collected at the anode. In a fast scintillation process all these electrons will arrive within a 10ns wide pulse, which can drive a 50mV signal into a 50Ω transmission line. Such signals readily trigger modern semi-conductor electronics such as a 10ns line receiver ⇒ bells, whistles, tape recorders etc.

The scintillation process is a bit more complicated than indicated above, if an atom has a good chance of de-exciting via photon emission, then similar neighboring atoms will readily absorb this light, possibly re-emitting it later. But as this cycle of emission and absorption is not 100% efficient, the light never reaches the surface of the material. Even most transparent materials are not transparent to their own scintillation!

A typical trick is to dope the material with some other atoms or molecules which are readily excited by the scintillation light, but de-excite via a cascade of lower energy photons (longer wavelength). The host material may then be transparent to the 'wave-shifted' photons.

There are 2 broad classes of scintillators: inorganic and organic. The inorganic scintillators were first used in 1903 (Crookes) in the form of ZnS (now used mainly on oscilloscope screens). Recent important inorganic scintillators include NaI (Th) (developed by Hofstadter at Princeton in 1947), BeF₂, SiO₂ and BaF. These materials have relatively high scintillation efficiency, but characteristic de-excitation times of 100-200ns. They are high T, high density materials and make excellent high energy photon detectors, via the electromagnetic cascade process discussed below. NaI yields about 1 scintillation photon for each 25 eV of energy deposited by a high energy particle.

Organic scintillators are mostly based on benzene ring compounds. The fuller the ring, the better the scintillation. One of the very best organic scintillators is anthracene. Most practical organic scintillators are made from toluene compounds dissolved in a suitable plastic.

Organic scintillators are 4-10 times less efficient than NaI, but have very rapid de-excitation times of 1-5 ns. As such they are among the fastest of all particle detectors presently available. Scintillation detectors are often used to provide a 'trigger' signal indicating when it is appropriate to sample a slower detector.

A good organic scintillation detector can yield about 200 photo-electrons for a 1cm thickness, taking into account ~10% efficiency of light collection, and 70% photo-electric efficiency. A typical scintillation counter costs $300.
Ionization Detectors

A large class of detectors is based on collection of charges which are ionized by the passage of a high energy particle thru a gas (or even liquids and solids nowadays). It is favorable to use a gas such as argon which has a low probability of 'attaching' the ionized electrons to neutral atoms (due to possible polar nature of the atoms or molecule, or formation of meta-stable molecular states...)

An ionization detector is basically a capacitor,

\[
\text{HIGH ENERGY PARTICLE} \rightarrow C \begin{cases} \text{ELECTRONS} \\ \text{VOLTAGE} \end{cases} \rightarrow \text{SIGNAL VOLTA GE}
\]

Under the influence of an applied voltage (not shown) the ionization electrons drift towards the anode plate (or wire) at typical velocity of 1 cm per microsecond. (The positive ions drift much more slowly and are ignored if the RC time constant is positive ion drift time) With the resistor to discharge the capacitor as shown, a voltage pulse is observed with characteristic fall time RC. This is made longer than the relevant electron drift time (rise time) so as to observe maximum signal, but otherwise as short as possible.

A parallel plate ionization chamber as sketched above is not useful for detecting single minimum ionization particles; too little charge is liberated.

Example: \(1\) cm of argon gas, \(p = 0.15\) atm \(1\) cm

\(dE/dx|_{\text{HIN}} = 1.5\) MeV/\(\mu\)A cm \(2\) (see table appended)

\(\%\) energy deposition \(= 2\) Kev.

\(\Rightarrow 60\) electrons/cm

This is too few for even the best low noise, high speed amplifiers to detect.

Parallel plate ion chambers find application in monitoring intensities of beams of \(10^6\) or more high energy particles, or in observation of low energy nuclear reactions in which recoil nuclei may yield \(10^6\) times minimum ionization...
An important variation, due to Rutherford & Geiger (1908), utilizes cylindrical geometry. As electrons drift towards the anode wire, they find themselves in stronger and stronger electric fields. If the potential difference in one mean free path of the drifting electron is greater than the ionization potential of the gas, then a gas amplification occurs. Secondary ionization takes place due to electron-atom collisions—leading to a chain reaction. Most of the secondary ionization occurs very close to the anode wire, so the rise time of the pulse can be quite short ~ 70 ns.

An important contribution to the chain reaction process comes from ultraviolet photons liberated during recombination of ionized atoms. These can travel at right angles to the electric field lines, and propagate the chain reaction along the entire anode wire—leading to the Geiger discharge. This gives a very large signal, readily detected with 1908 electronics. However, the chamber is rendered useless for quite some time after such a discharge. Nowadays, electronic amplifiers are sensitive enough that there is little need for the UV photon contribution to the gas amplification process. Typically, certain gases are added in small amounts to 'quench' the UV propagation. A useful discharge of gas amplification factor ~ 10^4 can be held to only 50-100 ns.

Example: a tube of radius 1 cm filled with Ar/NO.

A minimum ionizing particle with 1 cm path then leads to a discharge of 6 x 10^5 electrons, in say a 50 ns pulse. This leads to a 200 mV signal into 1000 ohms. (If RC constant short enough) well-designed integrated circuit amplifiers can process this signal further.

A common implementation of the cylindrical geometry is the multi-wire proportional chamber (MWPC). In these, there are 2 planes of cathode wires surrounding a plane of anode wires.

\[
\begin{array}{cccccccc}
  & & & & & \cdots \cdots \cdots \cdots & & \\
  & & & & \cdots \cdots \cdots \cdots \cdot & & \\
  & \cdots \cdots \cdots \cdots \cdot & & \cdots \cdots \cdots \cdots \cdot & & \\
  & & & & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \\
\end{array}
\]

Each anode wire is approximately in a cylindrical environment.

\[
\begin{array}{cccccccc}
  & & & & & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \\
\end{array}
\]
Typical wire spacing is 1-2 mm, while the anode wires are usually 0.008" diameter. Such a device gives 1 mm position resolution (if 1 mm wire spacing) over areas of 1-2 m². The cost is about $30 per wire.

Another variation is the drift chamber. Here one anode wire is placed within an elongated cell of cathode wires held at graded potentials.

\[ \text{charged particle} \quad \begin{align*} & V_0 \quad V_1 \quad V_2 \quad V_3 \quad V_4 \quad V_5 \quad V_6 \\ \end{align*} \quad \text{equipotentials} \]

The grading creates a uniform electric field over most of the cell, but near the anode with a cylindrical geometry obtains. Thus the ionization electrons drift at uniform velocity over most of the cell before initiating gas amplification close to the anode wire. By measuring the drift time (relative to a start signal supplied by a scintillation counter) the position of the primary ionization can be determined to an accuracy of 1-2 mm. Drift distances of up to 10 cm have been achieved. Many fewer wires are required in a drift-chamber than in a MWPC, but the cost of the time measurement electronics is higher (~ $200 per wire).

Coherent Radiation Detectors

An interesting type of particle detector involves the coherent response of a material to the passage of a high-energy charged particle. Examples are the Čerenkov counter and the transition radiation detector.

The Čerenkov effect occurs when the moving particle has velocity greater than the group velocity of light in the material; i.e., \( V > \frac{c}{n} \) where \( n \) is index of refraction. Then the coherent action of the material which normally produces waves that travel with velocity \( \frac{c}{n} \) can not keep up with the particle, leading to a 'shock wave' response.

\[ \cos \Theta = \frac{c}{nu} = \frac{1}{n^2} \]

The intensity of the radiation is quite low, being about 500 a.m.² photons/cm² in the visible light spectrum.
The Čerenkov effect can be utilized in a threshold detector, which responds to all particles of velocity \( v > c/n \).

**Particle**

**Spherical Mirror**

Optics exercise: Show that a line source of light which produces cones of rays of angle \( \Theta \), is focused to a point image by a spherical mirror.

The index \( n \) of a gas can be tuned by varying its density

\[
\frac{n^2 - 1}{n^2} \approx \frac{4\pi N}{M} \frac{N}{n^2}
\]

where \( N \) = number density of atomic electrons.

**Figure 2-5**. Pressure curve for \( C_4H \). As the helium pressure in the beam pipe is increased, the index of refraction is increased and heavier particles begin to cause Čerenkov radiation. As seen in the figure, a very good separation of pions and protons was possible. Kaons were counted as pions under normal running conditions.

A beam of particles prepared with the aid of various magnets might contain particles of several different masses, but all of the same momentum \( P \). Then velocity \( v = P/m \) depends on the particle type. If there are \( M \) particle types, then could be identified by type with the use of \( M-1 \) threshold Čerenkov counters, in a non-destructive way.

Differential counters use various optical baffles so as to collect light emitted only in a small angular range \( \Delta \Theta \) with respect to the particle’s direction. Then a single such device can identify exactly one particle type in a mixed beam of known momentum. This works well only in well collimated beams so that \( \Delta \Theta \) is sharply defined.

Transition radiation detectors utilize the change in shape of the electric field across the boundary between two dielectric media. The field of a moving charge is continually rearranging itself at the boundary, resulting in a pulse of radiation in addition to the static field component. This effect is much weaker than Čerenkov radiation, but the intensity varies as \( \gamma^4 \), where \( \gamma = 1/\sqrt{1-v^2/c^2} \).
TRANSITION RADIATION DETECTORS FIND THEIR MAIN APPLICATION IN THE IDENTIFICATION OF ELECTRONS NON-DESTRUCTIVELY (AS THEY HAVE THE HIGHEST Y FOR A GIVEN ENERGY). THE TRANSITION RADIATION IS MAINLY IN X-RAY FREQUENCIES, REQUIRING INTERESTING TECHNOLOGY FOR ITS OBSERVATION.

AVAILABLE ON REQUEST ARE A SET OF NOTES DERIVING THE INTENSITY AND SPECTRUM OF Y TRANSITION RADIATION WITHOUT USE OF BESSEL FUNCTIONS.

WE NOW CONSIDER A CLASS OF DETECTORS OF ELECTRONS, PROTONS AND CHARGED OR NEUTRAL NUCLEONS WHICH DEPEND ON THE ENERGETIC PARTICLE INITIATING A 'SHOWER' OF HIGH ENERGY INTERACTIONS. FIRST WE GIVE SOME DETAILS OF HIGH ENERGY ELECTRON AND PROTON INTERACTIONS WITH MATTER.

INTERACTIONS OF ELECTRONS WITH MATTER - BREMSSTRAHLUNG

AS ELECTRONS PASS THROUGH MATTER THEY SUFFER ΔE/Δx LOSSES AND MULTIPLE COULOMB SCATTERING AS DO NEUTRAL PARTICLES. HOWEVER, RELATIVISTIC ELECTRONS ARE EVEN MORE STRONGLY AFFECTED BY BREMSSTRAHLUNG - RADIATION OF PROTONS DURING COULOMB COLLISIONS WITH THE NUCLEUS. THE RADIATION RATE DEPENDS ON THE SQUARE OF THE ACCELERATION \( a^2 \). ONLY FOR THE RELATIVISTIC LIGHT ELECTRONS IS BREMSSTRAHLUNG A SIGNIFICANT LOSS MECHANISM.

WE GIVE A SEMI-CLASSICAL ARGUMENT. RECALL THE LARMOR FORMULA FOR RADIATION BY AN ACCELERATED CHARGE AT LOW VELOCITIES V \( \ll c \).

\[
\frac{dU}{dt} = \frac{2}{3} \frac{e^2}{c^3} a^2.
\]

WE NEED THE RESULT AS \( U \to c \), AND \( a \perp \vec{v} \) AS IN A COULOMB COLLISION.

A KEY FACT IS THAT \( \frac{dU}{dt} \) IS A RELATIVISTIC INVEARIANT: ENERGY \( U \) AND TIME \( t \) BOTH TRANSFORM AS THE TIME COMPONENT OF A 4-VECTOR.

Thus

\[
\left. \frac{dU}{dt} \right|_{\text{LAB}} = \frac{2}{3} \frac{e^2}{c^3} a^2 \text{ ELECTRONS REST FRAME}
\]

IN THE ELECTRON'S REST FRAME THE NUCLEUS MOVES WITH VELOCITY \( -\vec{v} \), SO ITS FIELD STRENGTH IS BOOSTED TO \( \frac{e}{b} \), AT IMPACT PARAMETER \( b \).

HENCE \( a = \frac{e^2}{b^2 m_e} \) AND

\[
\left. \frac{dU}{dt} \right|_{\text{LAB}} = \frac{2}{3} \frac{e^2}{b^4 m_e^2} \frac{e^2}{c^3}.
\]

THE PULSE OF RADIATION LASTS TIME \( \Delta t = 2b/c \) IN THE LAB, SO

\[
U = \frac{4}{3} \frac{e^2}{b^3} \left( \frac{e^2}{m_e c^2} \right)^2.
\]
The lab frequency spectrum is the Lorentz transform of the rest frame spectrum. In the rest frame the pulse lasts time $\Delta t^* = \frac{1}{8c}$ due to the Lorentz contraction of the field of the nucleus. Such a pulse has a flat frequency spectrum up to $\nu^* = \frac{1}{\Delta t^*} \sim \frac{1}{8c}$.

On transforming this spectrum to the lab, the maximum frequency is $\nu_{\text{max}} \sim \frac{1}{8} \nu^* \sim \frac{1}{8} \frac{1}{c}$.

Again a pulse leads to a flat energy spectrum

$$\frac{dU}{d\nu} = \frac{U_{\text{total}}}{\nu_{\text{max}}} = \frac{1}{3} \frac{Z^2 \alpha^2}{E^2 N_0 \frac{c}{2} (\frac{1}{k_c} \frac{1}{M_e c^2})^2}$$

$$\frac{dU}{d\nu}$$

We may convert this energy spectrum to a photon number spectrum by dividing by $h\nu$: $dN = \frac{dU}{h\nu}$

So $\frac{dN}{d\nu} = \frac{2}{3\pi} \frac{Z^2}{E^2} \frac{\alpha^2}{k_c} \frac{1}{N_0 \frac{c}{2} (\frac{1}{M_e c^2})^2} \frac{1}{h\nu}$

Note the characteristic $1/\nu$ fall off of the bremsstrahlung spectrum.

We can now estimate the radiation loss by an electron in crossing a slab of material of thickness $d\lambda$ ($\text{cm} / \text{cm}^2$)

$$\frac{dE}{d\lambda} = \frac{N_0}{A} \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{1}{2} \frac{b}{b_{\text{min}}} U_{\text{total}} (b)$$

In the relativistic limit we cannot immediately use our result

$$U_{\text{total}} = \frac{Z^2 \alpha^2}{E^2} \left( \frac{E}{M_e c^2} \right)^2$$

Instead we must note that the maximum energy photon possible is just the energy of the electron itself.

Thus $h\nu_{\text{max}} = E$. The subtlety is that the energy spectrum remains as derived above, only it is cut off at lower $\nu_{\text{max}}$.

Our revised estimate is then

$$U(b) = \frac{dU}{d\nu} \frac{E}{h} = \frac{2}{3\pi} \frac{Z^2}{E^2} \frac{\alpha^2}{k_c} \frac{1}{N_0 \frac{c}{2} (\frac{1}{M_e c^2})^2} \frac{E}{h\nu}$$

And $\frac{1}{E} \frac{dE}{db} = \frac{2}{3\pi} \frac{N_0}{A} \frac{Z^2}{E^2} \frac{k}{k_c} \frac{1}{b^2} \frac{1}{\nu_{\text{max}}^2} \frac{1}{b_{\text{min}}} \frac{b_{\text{max}}}{b_{\text{min}}} \left( \frac{1}{k_c} \frac{1}{M_e c^2} \right)^2$

Again $b_{\text{max}} \sim \text{radius of atom} = \frac{r_e}{\sqrt{2} \times \nu_3}$

$$h\nu_{\text{max}} \sim h \frac{\nu c}{b_{\text{max}}} = E \quad \text{electron}$$

While $b_{\text{min}}$ can be thought of as $\frac{k}{M_e c} = \frac{r_e}{k_c} = \text{electron Compton wavelength}$ (a bit of a stumble)
If we accept the above estimates, we find
\[ \frac{1}{E} \frac{dE}{dx} = \frac{4 \times 10^2}{A} \times r_0^2 \frac{2}{\sqrt{3}} \left( \frac{137}{\sqrt{2}} \right) \]

A detailed calculation, Bethe-Heitler (1934), shows
\[ \frac{1}{E} \frac{dE}{dx} = \frac{4 \times 10^2}{A} \times r_0^2 \frac{2}{\sqrt{3}} \left[ \ln \left( \frac{183}{x_0^2} \right) + \frac{1}{18} \right] \]

Then introduced the definition of the **radiation length** \( x_0 \)
on this basis:
\[ \frac{1}{E} \frac{dE}{dx} = \frac{1}{x_0} \quad \Rightarrow \quad E = E_0 e^{-x/x_0} \]

Electrons of all energies (once \( x \) large) radiate away \( x^{2/3} \)of their energy in one radiation length!

As far as interactions with matter are concerned, a high energy electron is one for which the radiation loss is greater than the ionization loss to atoms. This **critical energy** is about
\[ E_c = \frac{3 \pi}{2} \times \frac{600 \text{ MEV}}{2 r_0} \]

Comparing with p. 42.

---

Fig. 2.11.4. Fractional energy loss by collision, \( -\frac{1}{U} \left(\frac{dE}{dt}\right)_{\text{col}} \), and fractional energy loss by radiation, \( -\frac{1}{U} \left(\frac{dE}{dt}\right)_{\text{rad}} \), for electrons, per radiation length of air or lead.

That is, the **critical energy** is the energy loss to ionization when a particle traverses one radiation length.
Absorption of Photons Due to Pair Production

Low energy interactions of photons with matter are dominated by Compton scattering (= Thomson scattering at very low energies), and by the photo-electric effect. But if \( E_\gamma > 2m_e c^2 \), the photon can vanish, producing an \( e^+e^- \) pair.

The Bethe-Heitler process was mentioned in Lecture 3, and it was indicated that dimensional arguments lead to a cross section estimate:

\[
\sigma \approx \frac{2\pi^3}{M_e^2} \frac{\lambda_\gamma E_\gamma}{m_e} \approx \frac{2\pi^3}{M_e^2} \frac{\lambda_\gamma t}{E_\gamma} \frac{E_\gamma}{m_e^2}
\]

[Note also that the energy spectrum of the \( e^+e^- \) pair is essentially flat between 0 and \( E_\gamma \).

The detailed work of Bethe and Heitler showed that the pair production rate per \( g\gamma/cm^2 \) traversed is

\[
\frac{1}{9} \frac{4\pi N_0}{A} \frac{2\pi^3}{M_e^2} \lambda_\gamma \frac{183}{\sqrt{2}} \frac{1}{\sqrt{x_0}}
\]

(For \( E_\gamma > 300 \text{ MeV} \))

Thus high energy photons produce a pair typically before traversing one radiation length.

\[
N_\gamma = N_0 e^{-\frac{1}{9} \frac{4\pi N_0}{A} \frac{2\pi^3}{M_e^2} \lambda_\gamma \frac{183}{\sqrt{2}} \frac{1}{\sqrt{x_0}}} \]

Again, the pair production mechanism becomes more important than Compton scattering at about the critical energy defined on p 54.

Note however the slow rise in the pair production rate until the asymptotic value is achieved at about 1 GeV. (Below 1 GeV the term \( 1/\gamma E_\gamma/2m_e \) is still important.)

We note that charged particles can also produce electron-positron pairs by diagrams as sketched. This process, and bremsstrahlung are quite rare for heavy particles, but can lead to very large energy losses in a single Coulomb collision. Most of the large fluctuations in deposited energy are due to these processes.

Fig. 2.19.4. The total probability for radiation length of lead for Compton scattering (\( \mu_{\text{com}} \)), for pair production (\( \mu_{\text{pair}} \)) and for either effect (\( \mu_{\text{com+pair}} \)). For \( E < 10^8 \) ev, \( \mu_{\text{pair}} \) cannot be calculated with the formulas given in the text and a more accurate equation must be used (BHA34). From Rossi and Greisen (RB41.1).
ELECTROMAGNETIC SHOWERS

When high energy electrons or photons enter a block of material, they do not deposit their energy all at once. Rather, they initiate a cascade of secondary electrons and photons. Each electron radiates photons, and photons produce more electrons via pair production, until the energies drop below the critical energy. (What happens then?)

If the initial electron or photon energy is \( E \), we might then expect \( E/E_c \) electrons in the entire cascade (counting each electron). The number of generations in the shower is then

\[
N = \frac{E}{E_c} \quad \text{or} \quad N = \frac{4(E/E_c)}{ln 2}
\]

Each generation of radiation + pair production penetrates about 2 additional radiation lengths into the material.

Example: \( E = 100 \text{ GeV} \) material: lead, with \( E_c = 7 \text{ MeV} \)

Then \( N \approx 14 \approx 28 \text{ radiation lengths of shower} \).

In order to contain 99% of the energy of a 100 GeV shower takes over 20 \( X_0 \) in practice.

As the shower develops, some of the electrons and photons move transversely to the initial direction. But the entire cascade is reasonably well contained within a radius of \( 1X_0 \) for heavy materials.

A better approximation for the shower radius is

\[
R_{\text{show}} \approx 1X_0 \cdot \frac{Z}{E_c} \text{ in MeV}
\]

---

Fig. 5.1.1. Cloud-chamber picture of a large cascade shower. The plates across the chamber are lead, 1.27 cm thick. From C. Y. Chao.
STRONG INTERACTIONS OF HADRONS WITH MATTER

Particles containing quarks (hadrons) occasionally interact strongly with nuclei as they traverse a block of material. These interactions are rare but violent, and determine the large scale features of the penetration of hadrons in material.

The cross section for inelastic scattering of a high energy hadron on a nucleus of atomic number $A$ is about $40 \text{ mb} \cdot A^{1/3} \cdot A^{1/3}$ of quarks in hadron (why $A^{1/3}$ ?)

By 'inelastic' we mean a reaction in which the final state includes other particles than the original hadron and nucleus. This can happen in ways: additional hadrons (mostly $\pi$ mesons) are produced; and the nucleus may break up into various fragments. The passage of hadrons thru matter is little affected by elastic nuclear scattering which gives little net contribution to energy loss, and whose cross-section is only $\approx 5 \text{ mb} \cdot A^{1/3}$.

The inelastic cross section is quite small, so the high energy hadron has a fairly long mean free path. Values of this mean free path to a nucleus interaction length are listed for protons in the table appended to this lecture. (Can you verify these values starting from the cross section given above?) A representative value is the interaction length in solid iron of 7 inches.

The interaction length for neutrons is essentially the same as for protons. After its first inelastic nuclear interaction, a neutron can be observed via the secondary charged particles. The interaction length for $\pi^+ \& \pi^-$ mesons is about $3/2 \text{ that for protons}$, as their cross sections are only $1/3$ as large.

In an inelastic collision the number of secondary hadrons produced is a slow function of the interaction energy. The secondaries are mainly $\pi$ mesons $p^+, p^-; \pi^0, \pi^0 \rightarrow \gamma \gamma$, so that charged secondaries outnumber neutral $\gamma$ by about 2 to 1. The $\pi^0$ mesons decay quickly ($t \approx 10^{-16} \text{sec}$) via $\pi^0 \rightarrow \gamma \gamma$, which causes electromagnetic showers. For materials of atomic number greater than carbon, the radiation length is smaller than the interaction length, so the latter sets the scale of the hadronic cascade process.
The energy of the secondary particles has a Bremsstrahlung-like distribution: \( \frac{dE}{d\varepsilon} \sim \frac{dE}{d\varepsilon} \). The highest-energy secondary has energy \( \varepsilon/2 \) on average.

The secondary particles travel thru the material, losing energy slowly by ionization until they suffer strong nuclear interactions. In each nuclear interaction about 200 MeV is lost to nuclear binding energy \( j \), and in iron about 300 MeV is lost to ionization between each nuclear interaction. Thus the total number of nuclear interactions induced by a hadron of energy \( E \) is \( N = \frac{E}{500 \text{ MeV}} \), and the number of generations is \( u = \ln \frac{E}{242} \).

For example, a 100-GeV proton will typically produce a shower of \( 8 \times \ln 2 \) interaction lengths (\( \approx 2 \times \text{proton interaction length} \)) \( \approx 2 \text{ meters long} \).

A hadron calorimeter is a device to measure the total energy deposited in the shower described above. Because of the great mass of material required to contain a hadronic shower, calorimeters are almost always multilayer sandwiches of a dense material such as iron, and particle detectors such as scintillators or MWPCs. Several samples of the particle tracks are taken each interaction length. Good performance is achieved only if there is about 1 sample per radiation length to collect the 70% part of the shower. In effect, one tries to determine the total path length traveled by charged particles, and then use the minimum-ionization energy loss of \( \sim 1.5 \text{ MeV/cm} \) to estimate the total energy.

A serious difficulty is that the energy spent in nuclear bremsstrahlung (\( \sim 200 \text{ MeV/interaction} \)) is poorly sampled by this technique. Statistical fluctuations in this unsampled energy, 40% of the total on average, limit the energy resolution:

\[ N = \frac{E}{500 \text{ MeV nuclear interactions}} \sqrt{N} \text{ fluctuation} \]

\[ \frac{\Delta E}{E} = \frac{1}{N} = \frac{1}{10E/242} = \frac{1}{1000} \text{ in iron} \]

The use of uranium rather than iron leads to significantly smaller losses due to nuclear bremsstrahlung and resolutions of 50%/\( \varepsilon E \) are reported.

Despite this poor resolution, hadron calorimeters are becoming increasingly important in very high-energy experiments where knowledge of the total energy balance is important. For example, the discovery of the W boson via the decay \( W \to \ell \nu Y \) depended on showing that the neutrino escaped detection while carrying away energy in \( 1/2 \times 40 \text{ GeV} \).

A large multiparticle detector

On the next page is sketched the detector of the LEP-3 group which operates at CERN to observe \( \ell^+ \ell^- \to Z^0 \) and the subsequent decay of the \( Z^0 \). It emphasizes calorimetry of electrons, photons and hadrons, and magnetic measurement of muons. The detector is quite large.
### Particle Detectors, Absorbers, and Ranges (Cont'd)

Atomic and Nuclear Properties of Materials

| Material | Z | A | Nuclear collision cross section σ_{c}[barn] | Inelastic collision cross section σ_{i}[barn] | Nuclear interaction length λ_{n}[cm] | DE/dx cm^{-1} | Density ρ [g/cm^3] | Refractive index n |
|----------|---|---|------------------------------------------|------------------------------------------|-------------------------------|----------------|-------------------|------------------|------------------|
| H_2      | 1 | 1 | 0.0387                                  | 0.032                                    | 43.3                          | 4.12           | 0.292            | 63.05            | 0.0708           |
| D_2      | 1 | 2 | 0.073                                   | 0.052                                    | 45.7                          | 2.07           | 0.242            | 126.1            | 0.165            |
| He       | 2 | 4 | 0.138                                   | 0.109                                    | 49.9                          | 1.94           | 0.243            | 94.32            | 0.125(0.178)    |
| Li       | 3 | 6 | 0.216                                   | 0.158                                    | 53.8                          | 1.58           | 0.843            | 82.76            | 0.0534           |
| Be       | 4 | 9 | 0.288                                   | 0.199                                    | 58.5                          | 1.72           | 2.97             | 65.19            | 1.848            |
| C        | 6 | 12| 0.331                                   | 0.221                                    | 60.2                          | 1.78           | 4.03             | 42.70            | 2.256            |
| N_2      | 7 | 14| 0.379                                   | 0.262                                    | 61.4                          | 1.82           | 1.47             | 37.99            | 0.808(1.250)    |
| O_2      | 8 | 16| 0.420                                   | 0.288                                    | 63.2                          | 1.82           | 2.07             | 34.24            | 1.226(1.266)    |
| Ne       | 10| 20| 0.502                                   | 0.340                                    | 66.7                          | 1.73           | 2.09             | 28.94            | 1.07(0.90)      |
| Al       | 13| 26| 0.634                                   | 0.421                                    | 70.6                          | 1.62           | 4.37             | 24.01            | 2.70             |
| Ar       | 18| 39| 0.850                                   | 0.534                                    | 78.0                          | 1.51           | 2.11             | 19.55            | 1.40(1.78)      |
| Fe       | 26| 56| 1.103                                   | 0.703                                    | 83.3                          | 1.48           | 11.6             | 13.84            | 7.87             |
| Cu       | 29| 64| 1.232                                   | 0.782                                    | 85.6                          | 1.44           | 12.9             | 12.96            | 8.96             |
| Sn       | 50| 112| 1.191                                  | 1.002                                    | 165.5                         | 1.36           | 9.2              | 8.82             | 7.31             |
| W        | 74| 183| 1.267                                  | 1.649                                    | 150.3                         | 1.16           | 22.4             | 6.67             | 19.3             |
| Pb       | 82| 207| 1.960                                  | 1.776                                    | 193.7                         | 1.13           | 12.8             | 6.37             | 11.25            |
| U        | 92| 238| 3.378                                  | 1.983                                    | 199.3                         | 1.19           | 20.7             | 6.00             | 16.95            |

Air (20°C)  
H_2O: 62.0  
Shielding concrete: 67.4  
SiO_2 (quartz): 67.0

H_2 (bubble chamber 26°K): 43.3  
D_2 (bubble chamber 31°K): 45.7  
H-N mixture (90 mole percent): 65.0  
Propane (C_3H_8): 56.5  
Freon 113 (CF_2ClF_2): 76.8

Iford emulsion G3: 82.0  
LIF: 94.6  
BGO (Bi_4Ge_3O_12): 97.4  
Polyvinylstearic chloride (CH): 58.4  
Lucite, Plexiglas (C_2H_3O_2): 59.2  
Polyethylene (C_2H_4): 56.9  
Mylar (C_2H_5O_2): 60.2  
Borosilicate glass (Pyrex): 66.2  

CO_2: 62.4  
Methane CH_4: 54.7  
Isobutane C_3H_8: 56.3  
Freon 12 (CCl_2F_2): 68.1  
Silica Aerogel: 65.5

Spark or proportional chamber: 0.028%  
Sparkelectric: 0.020%  
Sparkelectric: 0.007%  

* Table revised March 1982 by J. Engler. For details, see Report KfK 8000, Kernforschungszentrum D 75 Karlsruhe, P.O. Box 3640, Federal Republic of Germany.

| b | Elastic scattering cross section σ_{elastic} for neutrons at 60-375 GeV from Roberts et al., Nucl. Phys., B159, 56 (1979). For protons and other particles, see Carroll et al., Phys. Rev. Lett. 38, 319 (1979); note that σ_{el}(p) = σ_{el}(n).
| c | Mean free path between collision (AT) or inelastic interaction (AI), calculated λ = 1/(NσA).
| d | For minimum-ionizing protons and pions from Barks and Barber, Tables of Energy Losses and Ranges of Heavy Charged Particles, NASA-SP-3013 (1964).
| e | For electrons see: Penetration of Charged Particles in Matter, NAS-NS39 (1964).
| f | From Y.S. Tsai, Rev. Mod. Phys. 46, 815 (1974).
| g | Values for solids, or the liquid phase at boiling point. Values in parentheses for gaseous phase STP (0°C, 1 atm.), except where noted. Refractive index given for sodium D line.
| h | For pure graphite, industrial graphite density may vary 2.1 - 2.3 g/cm^3.
| i | Standard shielding blocks, typical composition O_2 52%, Si 32.5%, Ca 6%, Na 1.5%, Fe 2%, Al 4% plus reinforcing iron bars. Attenuation length λ = 115 ± 5 g/cm^3, also valid for earth (typical p = 2.05) from CERN-LRL-RHEL Shielding exp., UCRL-17841 (1968).
| j | Density may vary within ±3%, depending on operating conditions.
| k | Values for typical working conditions with H_2 target: 50 mole percent, 29°K, 7 atm.
| l | Values for typical chamber working conditions: Propane ~57°C, 10-8 atm. Freon 13B1 ~28°C, 8-10 atm.
| m | Main components: 80% SiO_2 + 12% B_2O_3 + 5% Na_2O.
| o | (SiO_2) + 2n(H_2O) used in Cerenkov counters, p = density in g/cm^3. From M. Cantin et al., Nucl. Instr. Meth. 118, 177 (1974).
| p | Values for typical construction: 2 layers 50 μm Cu/Be wires, 8 mm gap, 60% argon, 40% isobutane or CO_2, 2 layers 50 μm Mylar/Acrylic foils.
PARTICLE DETECTORS, ABSORBERS, AND RANGES

A. DETECTOR PARAMETERS

In this section we give various parameters for common detectors. The quoted numbers represent at best an order of magnitude, and are useful only for preliminary design. A more detailed introduction to detectors can be found in "A Consumer’s Guide to Particle Detectors," by D.J. Miller, Rutherford Lab Report RL-76-072, July 1976.

A.1 Scintillators: Photon yield \( \approx 1/100 \text{ eV in plastic scintillator} \) and \( \approx 1/25 \text{ eV in NaI} \).

A.2 Čerenkov: 1/2 Half-angle \( \theta_c \) of cone aperture in terms of velocity \( \beta \) and index of refraction \( n \):

\[
\theta_c = \arccos \left( \frac{1}{\beta n} \right) = \left( 1 - \frac{1}{\beta^2 n^2} \right)^{1/2}.
\]

Threshold velocity:

\[
\beta_c = 1/n; \quad \gamma_c = \frac{1}{\sqrt{1 - \beta_c^2}}.
\]

Therefore, \( \beta \gamma = 1/\sqrt{\beta + 1}, \) where \( k = n - 1 \). Values of \( k \) for various commonly used gases are given as a function of pressure and wavelength in Ref. 4; for values at atmospheric pressure, see the Table of Atomic and Nuclear Properties, following.

Number of photons, \( N \), per cm:

\[
N = \frac{\phi}{c} \int \left( 1 - \frac{1}{\beta^2 n^2} \right) d\beta = \frac{\phi}{c} \left( \frac{1}{\beta_c^2} - \frac{1}{\beta^2} \right) d\beta
\]

\[
\approx 500 \text{ sin}^2 \theta_c \text{ cm}^{-1} \text{ (visible spectrum)}
\]

A.3 Photon Collection: In addition to the photon yield, one should take into account the light collection efficiency (\( \approx 10\% \) for typical 1-cm-thick scintillator), attenuation length (\( \approx 1 \text{ m} \) for typical scintillators), and quantum efficiency of the photomultiplier cathode (\( \approx 25\% \)).

A.4 Typical Detector Characteristics:

<table>
<thead>
<tr>
<th>Detector Type</th>
<th>Accuracy (rms)</th>
<th>Resolution Time</th>
<th>Dead Time</th>
</tr>
</thead>
</table>
| Bubble chamber       | \( \approx 50 \mu \) | \( \approx 1 \text{ ns} \) | \( \approx 1/20 \text{ s} \)
| Streamer chamber     | \( \pm 300 \mu \) | \( \approx 2 \mu \) | \( \approx 100 \text{ ns} \)
| Optical spark chamber| \( \pm 200 \mu \) | \( \approx 2 \mu \) | \( \approx 10 \text{ ns} \)
| Magnetoresistive     |                |                 |           |
| spark chamber        | \( \pm 500 \mu \) | \( \approx 2 \mu \) | \( \approx 10 \text{ ns} \)
| Proportional chamber | \( \pm 300 \mu \) | \( \approx 50 \text{ ns} \) | \( \approx 200 \text{ ns} \)
| Drift chamber        | \( \pm 50 \text{ to } 300 \mu \) | \( \approx 5 \text{ ps} \) | \( \approx 100 \text{ ns} \)
| Scintillator         | \( \pm 1 \mu \) | \( \approx 10 \text{ ns} \) |           |

\( ^a \) Multiple pulsing time.

\( ^b \) 60\% for high pressure.

\( ^c \) 30\% is for 1 mm pitch.

\( ^d \) Delay line cathode readout can give \( \pm 150 \mu \) parallel to anode wire.

\( ^e \) For two chambers.

A.5 Shower Detectors: Typical energy resolutions (FWHM) for incident electron in the 1 GeV range, E in GeV. For a fixed number of radiation lengths, FWHM in the last three detectors would be expected to be proportional to \( \sqrt{E} \) for \( t \approx \text{place thickness} \approx 0.2 \text{ radiation lengths} \).

<table>
<thead>
<tr>
<th>NaI (20 rad. lengths)</th>
<th>( ^{2}% )</th>
<th>( \frac{1}{\sqrt{E}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead Glass (14 rad. lengths)</td>
<td>( ^{3}% )</td>
<td>( \frac{10 - 12%}{{\sqrt{E}}} )</td>
</tr>
<tr>
<td>Lead-Liquid Argon (15.75 rad. lengths)</td>
<td>( ^{4}% )</td>
<td>( \frac{2}{\sqrt{E}} )</td>
</tr>
<tr>
<td>(42 cells: 1.1 mm lead, 2 mm liquid argon, 2.3 mm lead-G10, 2 mm liquid argon)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lead-Scintillator Sandwich (12.5 rad. lengths)</td>
<td>( ^{5}% )</td>
<td>( \frac{17%}{{\sqrt{E}}} )</td>
</tr>
<tr>
<td>(66 cells: 1 mm lead, 5 mm scintillator)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proportional Wire Shower Chamber (17 rad. lengths)</td>
<td>( ^{6}% )</td>
<td>( \frac{40%}{{\sqrt{E}}} )</td>
</tr>
<tr>
<td>(36 cells: 0.474 rad. length type-metal + Al, 9.5 mm 80% Ar + 20% CH4 gas)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A.6. Proportional Chamber Wire Loss: Typical wire tension \( T \), due to mechanical effects when the electrostatic repulsion of adjacent wires exceeds the restoring force of wire tension, is given by:

\[
T \leq \frac{3V^{1/2}}{\ell C}
\]

where \( V \), \( \ell \), and \( C \) are the wire spacing, length, and capacitance per unit length. An approximation to \( C \) for chamber half-gap \( t \) and wire diameter \( d \) (good for \( s \leq t \)) gives:

\[
V \leq \frac{5V^{1/2}}{\ell^{3/2}} \left\{ \frac{1}{\ell} - \frac{s}{\ell} \right\} \left( \frac{s}{t} \right) \left\{ \sinh \left( \frac{s}{a} \right) \right\} \right.
\]

where \( V \) is in kV, and \( T \) is in grams.

A.7 Proportional and Drift Chamber Potentials: Potential distributions and fields for an array of parallel line charges \( q \) (coul/m) along \( z \) and located at \( y = 0, x = 0, \pm a, \pm 2a, \ldots \), can usually be calculated with good accuracy from (MKSA):

\[
V(x,y) = -\frac{q}{4\pi \epsilon_0} \ell \left\{ 4 \left( \sin^2 \left( \frac{2\pi x}{a} \right) + \sinh^2 \left( \frac{2\pi y}{a} \right) \right) \right\}
\]

B. COSMIC RAY FLUXES

The fluxes of particles of different types depend on the latitude, their energy, and the conditions of measurement. Some typical sea-level values \( ^{13} \) for charged particles are given below:

<table>
<thead>
<tr>
<th>Component</th>
<th>Soft</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intensity</td>
<td>Component</td>
</tr>
<tr>
<td>( I_v )</td>
<td>( 0.8\times10^{-2} )</td>
</tr>
<tr>
<td>( J_1 )</td>
<td>( 1.3\times10^{-2} )</td>
</tr>
<tr>
<td>( J_2 )</td>
<td>( 1.7\times10^{-3} )</td>
</tr>
</tbody>
</table>

Very approximately, about 75% of all particles at sea-level are penetrating, and are muons. The absolute flux of protons at sea-level, in a momentum range 700-1100 MeV/c, is \( 1.5\times10^{-2} \text{ cm}^{-2} \text{ sec}^{-1} \text{ sterd}^{-1} \), or \( \sim 0.1% \) of all particles.

The muon flux at sea-level has a mean energy of 2 GeV and a differential spectrum falling as \( E^{-0.8} \), steepening smoothly to \( E^{-1.6} \) above a few TeV. The angular distribution is cos\( \theta \), changing to \( \sin \theta \) at energies above a TeV, where \( \theta \) is the zenith angle at production. The \( \pi^- \)/\( \mu^- \) ratio is 1.25-1.30. The mean energy of muons originating in the atmosphere is roughly 300 GeV at slant depths \( \sim 100 \text{ meters} \) and below.

C. PASSAGE OF PARTICLES THROUGH MATTER

C.1 Energy Loss Rates for Heavy Charged Projectiles: A heavy projectile (much more massive than an electron) of charge \( Z_{\text{inc}}e \), incident at speed \( \beta c \) (\( \beta \gg 1/132 \)) through a slowing medium, dissipates energy principally via interactions with the electrons of the medium. The mean rate of such energy loss per unit path length \( x \) may be written as:

\[
\frac{dE}{dx} = \frac{D Z_{\text{inc}}^2 e^2}{\beta c^2} \left( \frac{Z_{\text{inc}}}{\beta} \right)^2 \left\{ \ln \left( \frac{2m_0^2 \beta^4 c^2}{\ell} \right) - \beta^2 + \frac{3}{2} - \frac{c}{Z_{\text{med}}} \right\} \right\} \left( 1 + \nu \right).
\]
where $D = 4\pi N_e Z_e^2 m_e c^2 = 0.3070$ MeV cm$^2$/g (see Physical and Numerical Constants Table).

Here $Z_{\text{med}}$ and $\rho_{\text{med}}$ are the charge and mass numbers of the medium and $\rho_{\text{med}}$ is the mass density of the medium; $I$, $\beta$, $C$, and $\nu$ are phenomenological functions. Frequently, the values of $\beta$, $C$, and $\nu$ are negligibly small, the parameter $I$ characterizes the binding of the electrons of the medium. As a rule of thumb, we may estimate $I$ for an idealized medium as $I \approx 16$ ($Z_{\text{med}})^{0.9}$ eV when $Z_{\text{med}} > 1$. For realistic media the value of $I$ will vary at the 10% level from this estimate; for $H_2$, $I = 20.0$ eV. We may approximately treat media which are chemical mixtures or compounds by computing

$$\frac{d\sigma}{dx} \approx \frac{N}{\rho_{\text{med}}} \left( \frac{d\sigma}{dx} \right)_{\text{chem}} \rho_{\text{med}}^{\frac{1}{4}},$$

with $(d\sigma/dx)_{\text{chem}}$ appropriate to the $n^\text{th}$ chemical constituent (using $\rho_{\text{med}}^{(n)}$ as the partial density in the formula for $(d\sigma/dx)$).\(^{\text{17}}\)

The function $\delta$ represents the density effect upon the energy loss rate; it is non-negligible only for highly relativistic projectiles in dense media.\(^{\text{18}}\) For ultra-relativistic projectiles, $\delta$ approaches 2 $\delta$ as $+\infty$, where the value of the constant depends upon the density of the medium as well as its chemical composition.

The function $C$ represents shell corrections to the energy loss rate.\(^{\text{16}}\) These effects are non-negligible only for projectiles with speeds not much faster than the speeds of the fastest electrons bound in the medium.

The function $\nu$ represents corrections due to higher-order electrodynamics.\(^{\text{19}}\) These effects become important when $Z_{\text{med}}/\beta$ is comparable to 137. For relativistic unit-charge projectiles, $|\nu|$ is of the order of 1; positively charged projectiles lose energy more rapidly than do their charge conjugates.\(^{\text{19,20}}\)

For non-relativistic projectiles, our formulae above are inapplicable. At the very slowest speeds, total energy loss rates are believed to be proportional to $1/\beta$, rising to a peak at projectile speeds comparable to atomic speeds, after having passed through a smaller peak (due to the elastic Coulomb collisions with the nuclei of the slowing medium)\(^{\text{21}}\) at intermediate speeds. In some cases, energy loss rates depend significantly upon the relation of the projectile trajectory to the crystalline structure of the slowing medium.\(^{\text{22}}\)

For relativistic projectiles, $(d\sigma/dx)_{\text{chem}}$ falls rapidly with increasing $\beta$ until reaching a minimum around $\beta = 0.96$ (almost independent of medium), followed by a slow rise. Because of the density effect, the quantity in square brackets approaches $\delta$ when $\beta$ = constant for large $\gamma$.

The quantity $(d\sigma/dx)_{\text{med}}$ is the mean total energy loss via interactions with electrons of the medium in a layer of thickness $d\chi$. For any finite $d\chi$, Poisson fluctuations can cause the actual energy loss to deviate from the mean. For thin layers, the distribution is broad and skewed, being peaked below $(d\sigma/dx)d\chi$, and having a long tail toward large energy losses.\(^{\text{23}}\) Only for a very thick layer ($d\chi > 2m_e\beta^2/\gamma c^2$) will the distribution of energy losses become nearly Gaussian. The large fluctuations of the total energy loss rate from the mean are due to a small number of collisions involving large energy transfers. The fluctuations are greatly reduced for the so-called restricted energy loss rate, described in Section C.4.

C.2 Ionization Yields: Physicists frequently relate total energy loss to the number of ion pairs produced in the stopping medium. This relation becomes complicated for relativistic projectiles due to the wandering of energetic knock-on electrons whose ranges exceed the dimensions of the fiducial volume. For a qualitative appreciation of the non-locality of energy deposition by such moderately energetic knock-on electrons in various media, see Ref. 24. The mean energy loss per ion pair produced, $W$, is essentially constant for relativistic projectiles, but increases at slow projectile speeds.\(^{\text{25}}\) The numerical value of $W$ for gases can be surprisingly sensitive to trace amounts of various contaminants.\(^{\text{25}}\) Of course, in addition to the preceding effects, practical ionization yields may be greatly influenced by subsequent recombinations, etc.\(^{\text{26}}\)

C.3 Energetic Knock-On Electrons: For a relativistically point-like charge projectile, the production of high energy (kinetic energy $T > I$) electrons is given by (neglecting the spin of the electron):

$$\frac{d^2N}{dT \chi} = \frac{1}{2} \left[ \frac{Z_{\text{inc}}}{\rho_{\text{med}}} \right] \left[ \frac{Z_{\text{inc}}}{\beta} \right] \rho_{\text{med}} \frac{1}{T^{4}},$$

for $I < T \leq T_{\text{max}}$, where

$$T_{\text{max}} = \frac{2m_e\beta^2}{1 + 2y \frac{\nu_c}{M_{\text{inc}}} + \frac{\nu_c^2}{M_{\text{inc}}^2}},$$

where $M_{\text{inc}}$ is the mass of the incident particle, and all other quantities are as in Section C.1. This formula does not differ significantly from the precise result, incorporating spin effects, for any projectile (including $e^+$) in the restricted range $I < T < T_{\text{max}}$. More accurate formulae are available for various projectiles.\(^{\text{27,28}}\) Our formula is inaccurate for $T$ close to $I$; for $2I < T \leq 10I$, the $1/T^2$ dependence above becomes $T^{-3}$ with $3 < y < 5.29$.\(^{\text{29}}\)

C.4 Rates of Restricted Energy Loss for Relativistic Charged projectiles: The variability of energy loss for heavy projectiles is due primarily to the variability in the production of energetic knock-on electrons. Broadening and pair-production processes make this variability even greater for electrons than for heavy particles as projectiles (see, e.g., the figure "Fractional Energy Loss for $e^+$ and $e^−$ in Lead," following). If an instrument, such as a bubble chamber, is capable of isolating those high-energy-loss interactions, then it is appropriate to consider the rate of energy loss excluding them, i.e., a restricted energy loss rate. The mean energy loss rate via all collisions which have energy transfer $T$ such that $T < T_{\text{max}}$ is

$$\frac{d\sigma}{dx} \approx \frac{1}{2} \left[ \frac{Z_{\text{med}}}{\rho_{\text{med}}} \right] \rho_{\text{med}} \frac{1}{T^{4}} \left[ \frac{Z_{\text{inc}}}{\beta} \right] \left[ \frac{\nu_c}{M_{\text{inc}}} \right],$$

for $T < T_{\text{max}}$. Notice the overall factor of $1/2$.

The density effect causes the restricted energy loss rate to approach a constant, the Fermi plateau value, for the fastest projectiles.

C.5 Multiple Scattering through Small Angles: As a charged particle traverses a medium it is deflected by many small-angle elastic scatterings. The bulk of this deflection is due to elastic Coulomb scattering from the nuclei within the medium, hence the usual identification of multiple Coulomb scattering (note, however, that strong interactions do contribute to the total multiple scattering for hadronic projectiles). For both Coulomb and strong interactions, the Central Limit Theorem provides useful guidance in establishing the precise nature of the distribution of the total deflections resulting from multiple scattering. The true distribution is roughly Gaussian only for small deflection angles while it shows much greater probability for large-angle scatterings.\(^{\text{20}}\) An easier alternative which may suffice for non-critical applications would be to use a Gaussian approximation with the following width:

$$\theta = \frac{14.1}{\rho d} \sqrt{\frac{Z_{\text{inc}}}{T_{\text{R}}} \left[ 1 + \frac{1}{9} \log_{10} \left( \frac{T_{\text{R}}}{L_{\text{R}}} \right) \right] \text{ radians}},$$
where $p$, $\beta$, and $Z'_{\text{sc}}$ are the momentum (in MeV/c), velocity, and charge number of the incident particle, and $L/L_g$ is the thickness, in radiation lengths, of the scattering medium. $L_g$ for certain materials is given in the Table of Atomic and Nuclear Properties of Materials (following). The angle, $\theta_0$, is a fit to Molière theory, accurate to about 5% for $10^{-3} < L/L_g < 10$ except for very light elements or low velocity where the error is about 10 to 20%. In this Gaussian approximation, $\theta_0$ has the meaning

$$\theta_0 = \frac{\theta_{\text{plane}}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \theta_{\text{space}}.$$  

The non-projected (space) and projected (plane) angular distributions are given approximately by the Gaussian forms:

$$\frac{1}{2\pi^2} \exp \left( -\frac{\theta^2}{2\theta_0^2} \right) \, d\theta,$$

$$\frac{1}{\sqrt{2\pi} \theta_0} \exp \left( -\frac{\theta^2}{2\theta_0^2} \right) \, d\theta_{\text{plane}},$$

where $\theta$ is the deflection angle.

Other quantities are sometimes used to describe the amount of multiple Coulomb scattering: the auxiliary quantities $\phi_{\text{plane}}$, $y_{\text{plane}}$, and $s_{\text{plane}}$ (see the figure) obey:

$$y_{\text{plane}} = \frac{1}{\sqrt{3}} L \theta_{\text{plane}} = \frac{1}{\sqrt{3}} L \theta_0,$$

$$\phi_{\text{plane}} = \frac{1}{\sqrt{3}} \theta_{\text{plane}} = \frac{1}{\sqrt{3}} \theta_0,$$

and

$$s_{\text{plane}} = \frac{1}{4\sqrt{3}} L \theta_{\text{plane}} = \frac{1}{4\sqrt{3}} L \theta_0.$$

All the quantitative estimates in this section apply only in the limit of small $\theta_{\text{plane}}$ and in the absence of large-angle scatter.

C.6 Longitudinal Distribution of Electromagnetic Showers: A photon of energy $E$ (GeV) $\approx 0.1$ GeV converting in a semi-infinite medium produces an electromagnetic cascade whose intensity initially increases with depth and then falls off. The average number of $e^\pm$ with kinetic energy above 1.5 MeV, crossing a plane at a depth of $t$ radiation lengths from the beginning of the medium, in a material of atomic number $Z$, calculated using the Monte Carlo program EGS, can be fit by the empirical formula

$$N = N_0 r^{E_{\text{th}}} e^{-Bt},$$

where $N_0 = 5.51 \times 10^5 Z \frac{b}{a+1}(Z^{a+1})$, and $b = 0.634 - 0.0021 \times Z$ for $Z \geq 13, a = 2.0 - Z/340 + (0.664 - Z/340) \ln E$. For $Z = 13, a = 1.77 - 0.52 \ln E$. The maximum intensity, $N_{\text{max}}$, occurs at the depth $t = a/b$. The maximum error of the fit occurs in the vicinity of this depth and is less than 0.15 $N_{\text{max}}$. The integral of the tail, $\int N \, dt$ is fit to $1.5 \frac{a}{b}$ better than 2%. The total longitudinally-projected $e^\pm$ path length, $\int N \, dt = 5.51 \times 10^5 \sqrt{2}$, is less than the total $e^\pm$ path length due primarily to multiple Coulomb scattering.