Poynting’s Theorem with Magnetic Monopoles

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1 Problem

Deduce a version of Poynting’s theorem [1] in macroscopic electrodynamics supposing that magnetic charges (monopoles) exist in Nature. Discuss its relation to the Lorentz force law for magnetic charges.¹

2 Solution

When Heaviside first presented Maxwell’s equations in vector notation [3] he assumed that in addition to electric charge and current densities, \( \rho_e \) and \( \mathbf{J}_e \), there existed magnetic charge and current densities, \( \rho_m \) and \( \mathbf{J}_m \), although there remains no experimental evidence for the latter.² Maxwell’s equations for microscopic electrodynamics are then (in SI units)

\[
\nabla \cdot \varepsilon_0 \mathbf{E} = \rho_e, \quad \nabla \cdot \frac{\mathbf{B}}{\mu_0} = \rho_m, \quad -c^2 \nabla \times \varepsilon_0 \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \mu_0 + \mathbf{J}_m, \quad \nabla \times \frac{\mathbf{B}}{\mu_0} = \frac{\partial \varepsilon_0 \mathbf{E}}{\partial t} + \mathbf{J}_e, \quad (1)
\]

where \( c = 1/\sqrt{\varepsilon_0 \mu_0} \) is the speed of light in vacuum. In macroscopic electrodynamics we consider media that contain volume densities of electric- and Ampère magnetic-dipole moments, \( \mathbf{P}_e \) and \( \mathbf{M}_e \), respectively (often called the densities of polarization and magnetization). Supposing that magnetic charges exists, the media could also contain volume densities of (Gilbertian) electric- and magnetic-dipole moments, \( \mathbf{P}_m \) and \( \mathbf{M}_m \), respectively. These densities can be associated with bound charge and current densities, which together with the “free” charge and current densities \( \tilde{\rho}_e \), \( \tilde{\mathbf{J}}_e \), \( \tilde{\rho}_m \) and \( \tilde{\mathbf{J}}_m \) comprise the total charge and current densities, and are related by

\[
\rho_e = \tilde{\rho}_e - \nabla \cdot \mathbf{P}_e, \quad \mathbf{J}_e = \tilde{\mathbf{J}}_e + \frac{\partial \mathbf{P}_e}{\partial t} + \nabla \times \mathbf{M}_e, \quad (2)
\]

\[
\rho_m = \tilde{\rho}_m - \nabla \cdot \mathbf{M}_m, \quad \mathbf{J}_m = \tilde{\mathbf{J}}_m + \frac{\partial \mathbf{M}_m}{\partial t} - c^2 \nabla \times \mathbf{P}_m. \quad (3)
\]

It is customary in macroscopic electrodynamics to use versions of Maxwell’s equations in which only “free” charge and current densities appear. For this we introduce the fields³

\[
\mathbf{D}_e = \varepsilon_0 \mathbf{E} + \mathbf{P}_e, \quad \mathbf{H}_e = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}_e, \quad \mathbf{D}_m = \frac{\mathbf{E}}{\mu_0} - c^2 \mathbf{P}_m, \quad \mathbf{H}_m = \frac{\mathbf{B}}{\mu_0} + \mathbf{M}_m. \quad (4)
\]

¹Only in three spatial dimensions do the electric and magnetic fields of electric and magnetic charges have the same character, such that a single electric and a single magnetic field could describe the effects of both types of charges [2].

²Heaviside seems to have regarded magnetic charges as “fictitious,” as indicated on p. 25 of [4].

³Such “double” \( \mathbf{D} \) and \( \mathbf{H} \) fields were anticipated by Heaviside [5], who wrote \( \mathbf{H} \) for \( \mathbf{H}_e \) and \( \mathbf{h}_0 \) for \( \mathbf{H}_m \) near his eq. (88).
such that $D_e$ and $H_m$, and also $H_e$ and $D_m$, have similar forms, and

$$\nabla \cdot D_e = \tilde{\rho}_e, \quad \nabla \cdot H_m = \tilde{\rho}_m, \quad -\nabla \times D_m = \frac{\partial H_m}{\partial t} + \tilde{J}_m, \quad \nabla \times H_e = \frac{\partial D_e}{\partial t} + \tilde{J}_e. \quad (5)$$

where in the absence of magnetic charges $D_e$ and $H_e$ are the familiar fields $D$ and $H$.\(^4\)

In static situations with no “free” currents $\tilde{J}_e$ or $\tilde{J}_m$ the curls of both $D_m$ and $H_e$ are zero and these fields can be deduced from scalar potentials $V_e$ and $V_m$,

$$\nabla \times D_m = 0 \iff D_m = -\nabla V_e, \quad \nabla \times H_e = 0 \iff H_e = -\nabla V_m. \quad (6)$$

We can associate potential energies,

$$U_e = \mu_0 q_e V_e, \quad U_m = \mu_0 q_m V_m, \quad \quad (7)$$

with electric and magnetic “test” charges $q_e$ and $q_m$ in scalar potentials due to other charges. If those other charges are held fixed, the forces on the “test” charges can be written as

$$F_e = -q_e \nabla V_e = \mu_0 q_e D_m, \quad F_m = -q_m \nabla V_m = \mu_0 q_m H_e. \quad (8)$$

The magnetic version of eq. (8) was introduced by Poisson [8], and Maxwell [7] reflected this tradition by calling $H_e = H$ the magnetic force (per unit magnetic charge) and $B$ the magnetic induction. Note that eq. (8) holds in media with nonzero, static densities $P_e$, $P_m$, $M_e$ and $M_m$; the forces on charges inside static electromagnetic media are not $q_i E$ or $q_m B/\mu_0$.$^5,^6$ This contrasts with force calculations for the effective magnetic-charge density, $\rho_{m,\text{eff}} = -\nabla \cdot M_e$, which represent effects of Amp`erian currents, as discussed in Appendix A.

As noted in [10] and on p. 429 of [11], if a magnetic charge $q_m$ could be made to move around a loop some or all of which lies inside an Amp`erian magnetic material where $B$ does not equal $\mu_0 H_e$ (and hence $\nabla \times B$ is nonzero around the loop), then energy could be extracted from the system each cycle if the force were $q_m B/\mu_0$, and we would have a perpetual-motion machine. Similarly, if an electric charge $q_e$ could be made to move around a loop some or all of which lies inside a Gilbertian magnetic material where $E$ does not equal $D_m/\varepsilon_0$ (and hence $\nabla \times E$ is nonzero around the loop), then energy could be extracted from the system each cycle if the force were $q_e E$, and we would again have a perpetual-motion machine.

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\(^4\)The relation $B = \mu_0 (H + M)$ (or $B = H + 4\pi M$ in Gaussian units) seems to have been first introduced by W. Thomson in 1871, eq. (r), p. 401 of [6], and appears in sec. 399 of Maxwell’s *Treatise* [7].

\(^5\)The notion of the force on a static “test” charge inside a macroscopic medium is somewhat contradictory, in that the macroscopic fields are based on averages over volumes larger than atoms/molecules. People often suppose the test charge to be inside a cavity whose volume is at least as large as an atom/molecule, but then the magnitude of the force depends on the shape of the cavity. A more meaningful issue is the force on a “test” charge that moves through the medium, thereby sampling the microscopic fields in a way that can be well approximated in terms of the macroscopic fields. See also sec. 8 of [9].

\(^6\)In sec. 400 of [7], Maxwell noted that (in Gaussian units) the $H$ field inside a disk-shaped cavity with axis parallel to $B$ and $H$ inside a magnetic medium has $H_{cavity} = B_{cavity} = B_{medium} = H_{medium} + 4\pi M$, so that in this case one could say that the force on a magnetic charge $q_m$ in the cavity is $F_m = q_m H_{cavity} = q_m B_{medium}$. The led Maxwell to the characterization of $B$ as the “actual magnetic force,” which this author finds misleading.
The electromagnetic force on a moving electric charge $q_e$ and magnetic charge $q_m$, each with velocity $\mathbf{v}$, is, in microscopic electrodynamics,\(^7\)\(^8\)\(^9\)

\[
\mathbf{F}_e = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \mu_0 q_e (\mathbf{D}_m + \mathbf{v} \times \mathbf{H}_m),
\]

\[
\mathbf{F}_m = q_m \left( \mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E} \right) = \mu_0 q_m (\mathbf{H}_e - \mathbf{v} \times \mathbf{E}_e).\]

Consistency of the Lorentz force law with special relativity requires that either $\mathbf{E}$ and $\mathbf{B}$ or $\mathbf{D}_e$ and $\mathbf{H}_e$ or $\mathbf{D}_m$ and $\mathbf{H}_m$ appear in $\mathbf{F}_e$ and in $\mathbf{F}_m$ (see Appendix B). In macroscopic electrodynamics the Lorentz force law for the force density $\mathbf{f}$ on “free” charge and current densities takes the forms\(^10\)\(^11\)

\[
\mathbf{f}_e = \tilde{\rho}_e \mathbf{E} + \tilde{\mathbf{J}}_e \times \mathbf{B}, \quad \text{or} \quad \mu_0 (\tilde{\rho}_e \mathbf{D}_m + \tilde{\mathbf{J}}_e \times \mathbf{H}_m),
\]

\[
\mathbf{f}_m = \tilde{\rho}_m \mathbf{B} - \frac{\tilde{\mathbf{J}}_m}{c^2} \times \mathbf{E}, \quad \text{or} \quad \mu_0 (\tilde{\rho}_m \mathbf{H}_e - \tilde{\mathbf{J}}_m \times \mathbf{D}_e).
\]

It has been verified that $\mathbf{B}$ not $\mathbf{H}$ deflects high-energy electrically charged particles as they pass through magnetized iron (with no magnetic charges) [29], which confirms the first form of $\mathbf{f}_e$ in eq. (11).\(^12\) See also [32, 33, 34]. However, we need a theoretical argument to settle this issue for the case of Gilbertian magnetic charges.

Poynting’s argument [1] relates the rate of work done by electromagnetic fields on “free” electric and magnetic currents to both flow of energy and to rate of change of stored energy. This argument has delicacies of interpretation, discussed, for example, in sec. 2.19 of [35]. The density of the time rate of change of work on ("free") electric currents is, from eq. (11),

\[
\frac{d\tilde{w}_e}{dt} = \mathbf{f}_e \cdot \mathbf{v}_e = \tilde{\mathbf{J}}_e \cdot \mathbf{E}, \quad \text{or} \quad \mu_0 \tilde{\mathbf{J}}_e \cdot \mathbf{D}_m.
\]

Thus, either

\[
\frac{d\tilde{w}_e}{dt} = \tilde{\mathbf{J}}_e \cdot \mathbf{E} = \mathbf{E} \cdot \left( \nabla \times \mathbf{H}_e - \frac{\partial \mathbf{D}_e}{\partial t} \right)
\]

\(^7\)Lorentz advocated the form $\mathbf{F}_e = \mu_0 q_e (\mathbf{D}_e + \mathbf{v} \times \mathbf{H}_e)$ in eq. (V), sec. 12 of [12], although he seems mainly to have considered its use in vacuum. See also eq. (23) of [13]. That is, Lorentz considered $\mathbf{D}_e$ and $\mathbf{H}_e$, rather than $\mathbf{E}$ and $\mathbf{B}$, to be the macroscopic electromagnetic fields.

\(^8\)It is generally considered that Heaviside first gave the Lorentz force law (9) for electric charges in [14], but the key insight is already visible for the electric case in [3] and for the magnetic case in [15]. The form of $\mathbf{F}_m$ in terms of $\mathbf{B}$ and $\mathbf{E}$ is implicit in eq. (7) of [16] and explicit in sec. 28B of [17]. See also [18, 19].

\(^9\)For the macroscopic equations to appear as in eq. (5), as given, for examples, in sec. 7.3.4 and prob. 7.60 of [20], the Lorentz force law must have the form (10) for magnetic charges. One can also redefine the strength of magnetic charges, $\rho_m \rightarrow \rho_m / \mu_0$, $\mathbf{J}_m \rightarrow \mathbf{J}_m / \mu_0$, which leads to the forms given, for example, in sec. 6.11 of [21]. These alternative definitions echo a debate initiated by Clausius in 1882 [22].

\(^10\)A subtlety is that the field $\mathbf{B}$ in the first form of eq. (11) is not the total field, but rather the field at the location of the free current that would exist in its absence. See, for example, [23], especially sec. 4.

\(^11\)In 1908-10, Einstein argued that the Lorentz force law should take the form $\mathbf{f}_e = \mu_0 (\tilde{\rho}_e \mathbf{D}_e + \tilde{\mathbf{J}}_e \times \mathbf{H}_e)$ inside materials [24, 25], perhaps based on a misunderstanding discussed in [26], or that discussed in sec. 2.3.1 of [27]. This misunderstanding underlies the recent “paradox” of Mansuripur [28].

\(^12\)The magnetization of materials such as iron depends on the character of the magnetic moment of electrons. An argument due to Fermi [30] that the hyperfine interaction depends on the magnetic field at the origin, and so can distinguish between Ampèreian and Gilbertian moments of “nuclei.” For the case of positronium ($e^+ e^-$) the data imply that the moment of the electron is Ampèreian, as discussed in [31].
Poynting’s theorem still applies if magnetic charges exist. That is, the Poynting vector, which is the same form as if only electric charges exist, and hence the usual version (as also required not to have magnetic perpetual-motion machines).

\[
\text{where the total current } J_m \text{ rather than the “free” current } \tilde{J}_m \text{ appears in the last line, or}
\]

\[
\frac{d\tilde{w}_e}{dt} = \mu_0 \tilde{J}_e \cdot \dot{D}_m = \mu_0 D_m \cdot \left( \nabla \times H_e - \frac{\partial D_e}{\partial t} \right)
\]

\[
= -\mu_0 \nabla \cdot (D_m \times H_e) + \mu_0 H_e \cdot \nabla \times D_m - \mu_0 D_m \cdot \frac{\partial D_e}{\partial t}.
\]

\[
= -\mu_0 \nabla \cdot (D_m \times H_e) - \mu_0 D_m \cdot \frac{\partial D_e}{\partial t} - \mu_0 H_e \cdot \frac{\partial H_m}{\partial t} - \mu_0 \tilde{J}_m \cdot H_e. \quad (15)
\]

Similarly, for “free” magnetic currents in macroscopic electrodynamics we have, from eq. (12),

\[
\frac{d\tilde{w}_m}{dt} = f_m \cdot v_m = \tilde{J}_m \cdot B \text{ or } 
\mu_0 \tilde{J}_m \cdot H_e. \quad (16)
\]

A requirement of simplicity of Poynting’s theorem when magnetic charges are included favors that the time rate of change of the work done on “free” electric and magnetic currents be the second forms in eqs. (13) and (16), and that the Lorentz force law on macroscopic electric and magnetic charge and current densities be

\[
f_e = \tilde{\rho}_e \dot{E} + \tilde{J}_e \times \dot{B}, \quad (17)
\]

\[
f_m = \mu_0 (\tilde{\rho}_m H_e - \tilde{J}_m \times \dot{D}_e) = \mu_0 \tilde{\rho}_m (H_e - v_m \times \dot{D}_e), \quad (18)
\]

(as also required not to have magnetic perpetual-motion machines).\(^{13,14}\)

Then,

\[
\frac{d\tilde{w}}{dt} = \frac{d\tilde{w}_e}{dt} + \frac{d\tilde{w}_m}{dt} = -\mu_0 \nabla \cdot (D_m \times H_e) - \mu_0 D_m \cdot \frac{\partial D_e}{\partial t} - \mu_0 H_e \cdot \frac{\partial H_m}{\partial t} \equiv - \left( \nabla \cdot S + \frac{\partial u}{\partial t} \right). \quad (19)
\]

which is the same form as if only electric charges exist, and hence the usual version of Poynting’s theorem still applies if magnetic charges exist. That is, the Poynting vector,

\[
S = \mu_0 D_m \times H_e \quad \text{(all media),} \quad (20)
\]

\(^{13}\)The form (18) is also affirmed in [36] via considerations of a magnetic current in a “wire” surrounded by a dielectric medium. The issues here are somewhat different from those for the force on individual moving charges, but are similar to those considered in [37] for an electrical current in a wire inside a magnetic medium.

\(^{14}\)It is argued in [38] that a slowly moving magnetic charge perturbs electric polarization of a dielectric medium in such a way that the velocity-dependent force is \(-q_m v \times \epsilon_0 E\), where \(E = D/\epsilon\) is the electric field in the absence of the moving magnetic charge. The argument of [38] seems to this author to be a variant of sec. 400 of [7] in which it is supposed that the charge resides in a “cavity” whose surface details affect the fields experienced by the charge. Such arguments assume that the charge occupies a volume at least equal to one atom/molecule of the medium, which might have seemed reasonable to Maxwell but is not consistent with our present understanding of the size of elementary charges. The results of [29] show that a moving electric charges does not create a “cavity” inside a magnetic medium wherein the average \(B\) field differs from the macroscopic average \(B\) field in the absence of the charge. We infer that a moving magnetic charge would experience an average \(\dot{D}\) inside a dielectric medium equal to the macroscopic average \(\dot{D}\) field in the absence of the charge.
is interpreted as describing the flow of energy in the electromagnetic field, and for linear media in which $D_e$ and $D_m$ are both proportional to $E$, and $H_e$ and $H_m$ are both proportional to $B$, the density $u$ of stored energy associated with the electromagnetic fields is

$$u = \mu_0 \frac{D_e \cdot D_m + H_e \cdot H_m}{2} \quad \text{(linear media)}.$$  \hspace{1cm} (21)

That Poynting’s theorem retains its usual form when magnetic charges are present is discussed by Heaviside in sec. 19 of [5]. That the form of the Lorentz force law for magnetic charge and current densities is given by eqs. (17)-(18) is consistent with Heaviside’s argument; for example, his eq. (88), but is not explicitly stated. See also sec. 50, p. 49 of [4].

A peculiar argument that the “ordinary” form of Poynting’s theorem implies the existence of magnetic charges is given in sec. 7.10 of [39]; thus misunderstanding is clarified in [40].

The extension of Poynting’s theorem to momentum flow, with the implication that $\mu_0 D_e \times H_m$ is the density of stored momentum, as argued by Minkowski [41], remains valid if the Lorentz force law for magnetic charges is given by eqs. (17)-(18), but not for other forms, as discussed in sec. V of [38]. See also Appendix C.

In a search for an isolated magnetic charge $q_m$ in media that otherwise contain only electric charges and currents, $D_e \rightarrow D$, $D_m \rightarrow E/\mu_0$, $H_e \rightarrow H$, $H_m \rightarrow B/\mu_0$, the Lorentz force law reduces to

$$F_e = q_e (E + v_e \times B), \quad F_m = \mu_0 q_m (H - v_m \times E).$$  \hspace{1cm} (22)

### A Appendix: Effective Magnetic Charge Density

$$\rho_{m,\text{eff}} = -\nabla \cdot M$$

So far as is presently known, magnetic charges do not exist, and all magnetic effects can be associated with electrical currents, as first advocated by Ampère [42].\(^\text{15}\) For materials with magnetization density $M_e = M$ the associated (macroscopic) electrical current density is

$$J_e = \nabla \times M,$$  \hspace{1cm} (23)

and on the surface of such materials there is the surface current density

$$K_e = \hat{n} \times M,$$  \hspace{1cm} (24)

where $\hat{n}$ is the outward unit vector normal to the surface.

Alternatively, we can suppose the magnetization is associated with densities of effective magnetic charges. Some care is required to use this approach, since a true (Gilbertian) magnetic charge density $\rho_m$ would obey $\nabla \cdot B = \mu_0 \rho_m$ as in eq. (1), and the static force density on these charges would be $F_m = \mu_0 \rho_m H_e$. However, in Nature $\nabla \cdot B = 0 = \nabla \cdot \mu_0 (H + M)$, so we can write

$$\nabla \cdot H = -\nabla \cdot M = \rho_{m,\text{eff}},$$  \hspace{1cm} (25)

\(^{\text{15}}\)For discussion of the experimental evidence that “permanent” magnetization is Ampérien, see [27].
and identify

\[ \rho_{m,\text{eff}} = -\nabla \cdot \mathbf{M} \]  

(26)
as the volume density of effective (Ampèrian) magnetic charges.

Inside linear magnetic media, where \( \mathbf{B} = \mu \mathbf{H} \), the Maxwell equation \( \nabla \cdot \mathbf{B} = 0 \) then implies that \( \rho_{m,\text{eff}} = 0 \). However, a surface density \( \sigma_{m,\text{eff}} \) of effective magnetic charges can exist on an interface between two media, and we see that Gauss’ law for the field \( \mathbf{H} \) implies that

\[ \sigma_{m,\text{eff}} = (\mathbf{H}_2 - \mathbf{H}_1) \cdot \mathbf{n}, \]  

(27)
where unit normal \( \mathbf{n} \) points across the interface from medium 1 to medium 2. The magnetic surface charge density can also be written in terms of the magnetization \( \mathbf{M} = \mathbf{B}/\mu_0 - \mathbf{H} \) as

\[ \sigma_{m,\text{eff}} = (\mathbf{M}_1 - \mathbf{M}_2) \cdot \mathbf{n}, \]  

(28)
since \( \nabla \cdot \mathbf{B} = 0 \) insures that the normal component of \( \mathbf{B} \) is continuous at the interface.

The force on the surface density of effective magnetic charges is

\[ \mathbf{F} = \sigma_{m,\text{eff}} \mathbf{B}, \]  

(29)
since the effective magnetic charges, which are a representation of effects of electrical currents, couple to the magnetic field \( \mathbf{B} \), as in eq. (9).

16 Equation (29) is in agreement with prob. 5.20 of [21], recalling the different convention for factors of \( \mu_0 \) used there. However, the Coulomb Committee in their eq. (1.3-4) [43], and Jefimenko in his eq. (14-9.9a,b) [44], recommends that the field \( \mathbf{H}/\mu_0 \) be used rather than \( \mathbf{B} \) when using the method of effective magnetic charges, which would imply a force \( \mu_0/\mu \) times that of eq. (29) for linear media.

The total force on a linear medium is, in this view, the sum of the force on the conduction current plus the force on the effective magnetic surface charges. Care is required to implement such a computation of the force, as discussed in [37], where eq. (29) is affirmed by example.

The key result of this Appendix is that while “true” (Gilbertian, and nonexistent in Nature) magnetic charges \( \mathbf{p} \) obey the force law \( \mathbf{F}_{\text{m,true}} = \mu_0 \mathbf{p} \mathbf{H} \), the effective (Ampèrian) magnetic charges (which are a representation of effects of electrical currents) obey \( \mathbf{F}_{\text{m,eff}} = \mathbf{p}_{\text{eff}} \mathbf{B} \).

For “effective” Ampèrian magnetic charges the magnetic fields obey \( \nabla \cdot \mathbf{B}/\mu_0 = 0 \) and \( \nabla \cdot \mathbf{H}_e = \rho_{m,\text{eff}} \) inside magnetic materials, while for “true” Gilbertian magnetic charges the fields obey \( \nabla \cdot \mathbf{B}/\mu_0 = \rho_{m,\text{true}} \) and \( \nabla \cdot \mathbf{H}_m = 0 \) inside magnetic materials where there are no “free,” “true” magnetic charges. Hence, the roles of \( \mathbf{B}/\mu_0 \) and \( \mathbf{H} \) are reversed in magnetic materials that contain “true” or “effective” magnetic charges. We illustrate this below for the fields of a uniformly magnetized sphere.

### A.1 Fields of a Uniformly Magnetized Sphere

In this subappendix we deduce the static magnetic fields associated with uniform spheres of radius \( a \) with either uniform Gilbertian magnetization density \( M_m \) or uniform Ampèrian (effective) magnetization density \( M_e \).
A.1.1 Uniform Ampèrian Magnetization Density $M_e$

The total magnetic moment of the sphere is

$$\mathbf{m}_e = \frac{4\pi M_e a^3}{3}. \quad (30)$$

We speed up the derivation by noting that the fields inside the sphere are uniform, and the fields outside the sphere are the same as those of a point magnetic dipole of strength $\mathbf{m}_e$,

$$\frac{\mathbf{B}(r > a)}{\mu_0} = \mathbf{H}_e(r > a) = \mathbf{H}_m(r > a) = \frac{3(\mathbf{m}_e \cdot \hat{r})\hat{r} - \mathbf{m}_e}{4\pi r^3} = \frac{M_e a^3(2 \cos \theta \hat{r} - \sin \theta \hat{\theta})}{3r^3}, \quad (31)$$

in a spherical coordinate system with origin at the center of the sphere and $z$-axis parallel to $\mathbf{M}_e$.

To characterize the fields inside the sphere, we note use the method of effective magnetic charges (Appendix A). Since $\mathbf{M}_e$ is constant inside the sphere, there is no net effective magnetic charge density there, $\rho_{e,\text{eff}}(r < a) = -\nabla \cdot \mathbf{M}_e(r < a) = 0$, while there is a nonzero surface density of effective magnetic charge,

$$\sigma_{e,\text{eff}}(r = a) = \mathbf{M}_e \cdot \hat{r} = M_e \cos \theta. \quad (32)$$

The boundary condition on the magnetic field $\mathbf{H}_e$ at the surface of the sphere is that

$$H_{e,r}(r = a^+) - H_{e,r}(r = a^-) = \sigma_{e,\text{eff}}(r = a), \quad (33)$$

and hence,

$$H_{e,r}(r = a^-) = H_{e,r}(r < a) = H_e(r < a) \cos \theta = H_{e,r}(r = a^+) - \sigma_{e,\text{eff}}(r = a) = \frac{2M_e \cos \theta}{3} - M_e \cos \theta = -\frac{M_e \cos \theta}{3}, \quad (34)$$

$$\mathbf{H}_e(r < a) = -\frac{M_e}{3}, \quad \mathbf{H}_m(r < a) = \frac{\mathbf{B}(r < a)}{\mu_0} = \mathbf{H}_e(r < a) + \mathbf{M}_e(r < a) = \frac{2M_e}{3}. \quad (35)$$

The result (35) for $\mathbf{B}/\mu_0$ implies that the magnetic field for the idealization of a “point,” “effective” (Amperian) magnetic dipole $\mathbf{m}_e$ would be

$$\frac{\mathbf{B}}{\mu_0} = \frac{3(\mathbf{m}_e \cdot \hat{r})\hat{r} - \mathbf{m}_e}{4\pi r^3} + \frac{2\mathbf{m}_e}{3} \delta^3(\mathbf{r}). \quad (36)$$

A.1.2 Uniform Gilbertian Magnetization Density $M_m$

The total magnetic moment of the sphere for this case is

$$\mathbf{m}_m = \frac{4\pi M_m a^3}{3}. \quad (37)$$
As in sec. D.1.1, we speed up the derivation by noting that the fields inside the sphere are uniform, and the fields outside the sphere are the same at those of a point magnetic dipole of strength \( m_m \),

\[
\frac{B(r > a)}{\mu_0} = H_e(r > a) = H_m(r > a) = \frac{3(m_m \cdot \hat{r})\hat{r} - m_e}{4\pi r^3} = \frac{M_m a^3 (2 \cos \theta \hat{r} - \sin \theta \hat{\theta})}{3r^3}. \quad (38)
\]

To characterize the fields inside the sphere, we note use the method of effective magnetic charges (Appendix A). Since \( M_m \) is constant inside the sphere, there is no net true magnetic charge density there, \( \rho_m(r < a) = -\nabla \cdot M_m(r < a) = 0 \), while there is a nonzero surface density of true magnetic charge,

\[
\sigma_m(r = a) = M_m \cdot \hat{r} = M_m \cos \theta. \quad (39)
\]

The boundary condition on the magnetic field \( B \) at the surface of the sphere is that

\[
B_r(r = a^+) - B_r(r = a^-) = \mu_0 \sigma_m(r = a), \quad (40)
\]

and hence,

\[
\frac{B_r(r = a^-)}{\mu_0} = \frac{B_r(r < a)}{\mu_0} = \frac{B(r < a) \cos \theta}{\mu_0} = \frac{B_r(r = a^+)}{\mu_0} - \sigma_r(r = a) = \frac{2M_m \cos \theta}{3} - M_m \cos \theta = -\frac{M_m \cos \theta}{3}, \quad (41)
\]

\[
\frac{B(r < a)}{\mu_0} = H_e(r < a) = -\frac{M_m}{3}, \quad H_m(r < a) = \frac{B(r < a)}{\mu_0} + M_e(r < a) = \frac{2M_e}{3}. \quad (42)
\]

Comparing with eqs. (34)-(35) we see that the roles of \( B \) and \( H \) are reversed in the case of uniform true and effective magnetization. In particular, the sign of \( B \) inside the magnetized sphere is opposite for the cases of Ampèrian and Gilbertian magnetization, although \( B \) is the same outside the sphere in the two cases.\(^{17}\)

The result (42) for \( B/\mu_0 \) implies that the magnetic field for the idealization of a “point,” “true” (Gilbertian) magnetic dipole \( m_m \) would be

\[
\frac{B}{\mu_0} = \frac{3(m_m \cdot \hat{r})\hat{r} - m_m}{4\pi r^3} - \frac{m_m}{3} \delta^3(r). \quad (43)
\]

\section{Appendix: Lorentz Transformations of the Fields}

The various electromagnetic fields can be embedded in antisymmetric 4-tensors (six vectors) that obey Lorentz transformations. The microscopic fields \( E \) and \( B \) can be written as

\(^{17}\)For the case of a cylinder with uniform transverse magnetization, see [45], where the interior \( B \) field is equal and opposite for Ampèrian and Gilbertian magnetization.
components of the tensor $\mathbf{F}$, and of its dual $\mathbf{F}^\star$ obtained by the transformation $\mathbf{E} \rightarrow c\mathbf{B}$, $c\mathbf{B} \rightarrow -\mathbf{E}$,

$$
\mathbf{F} = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -cB_z & cB_y \\
-E_y & cB_z & 0 & -cB_x \\
-E_z & -cB_y & cB_x & 0
\end{pmatrix}, \quad \mathbf{F}^\star = \begin{pmatrix}
0 & cB_x & cB_y & cB_z \\
-cB_x & 0 & E_z & -E_y \\
-cB_y & -E_z & 0 & E_x \\
-cB_z & E_y & -E_x & 0
\end{pmatrix}, \quad (44)
$$
such that the fields in the ′ frame where the frame of eq. (44) has velocity $\mathbf{v}$ are

$$
\mathbf{E}' = \gamma \left( \mathbf{E} - \frac{\mathbf{v} \times c\mathbf{B}}{c} \right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{E})\hat{\mathbf{v}}, \quad (45)
$$
$$
c\mathbf{B}' = \gamma \left( c\mathbf{B} + \frac{\mathbf{v} \times \mathbf{E}}{c} \right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot c\mathbf{B})\hat{\mathbf{v}}. \quad (46)
$$

The densities of Ampérian electric and magnetic dipole moments $\mathbf{P}_e$ and $\mathbf{M}_e$ comprise the tensor $\mathbf{P}_e$, as first noted by Lorentz [46],

$$
\mathbf{P}_e = \begin{pmatrix}
0 & P_{e,x} & P_{e,y} & P_{e,z} \\
-P_{e,x} & 0 & M_{e,z}/c & -M_{e,y}/c \\
-P_{e,y} & -M_{e,z}/c & 0 & M_{e,x}/c \\
-P_{e,z} & M_{e,y}/c & -M_{e,x}/c & 0
\end{pmatrix}, \quad (47)
$$
such that the fields in the ′ frame where the frame of eq. (47) has velocity $\mathbf{v}$ are

$$
\mathbf{P}'_e = \gamma \left( \mathbf{P}_e + \frac{\mathbf{v} \times \mathbf{M}_e}{c} \right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{P}_e)\hat{\mathbf{v}}, \quad (48)
$$
$$
\frac{\mathbf{M}'_e}{c} = \gamma \left( \frac{\mathbf{M}_e}{c} - \frac{\mathbf{v} \times \mathbf{P}_e}{c} \right) - (\gamma - 1) \left( \hat{\mathbf{v}} \cdot \frac{\mathbf{M}_e}{c} \right) \hat{\mathbf{v}}, \quad (49)
$$

The macroscopic fields $\mathbf{D}_e = \epsilon_0 \mathbf{E} + \mathbf{P}_e$ and $\mathbf{H}_e = \mathbf{B}/\mu_0 - \mathbf{M}_e$ can be written as components of the tensor $\mathbf{G}_e$,

$$
\mathbf{G}_e = \epsilon_0 \mathbf{F} + \mathbf{P}_e = \begin{pmatrix}
0 & D_{e,x} & D_{e,y} & D_{e,z} \\
-D_{e,x} & 0 & -H_{e,z}/c & H_{e,y}/c \\
-D_{e,y} & H_{e,z}/c & 0 & -H_{e,x}/c \\
-D_{e,z} & -H_{e,y}/c & H_{e,x}/c & 0
\end{pmatrix}, \quad (50)
$$
such that the fields in the ′ frame where the frame of eq. (50) has velocity $\mathbf{v}$ are

$$
\mathbf{D}'_e = \gamma \left( \mathbf{D}_e - \frac{\mathbf{v} \times \mathbf{H}_e}{c} \right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{D}_e)\hat{\mathbf{v}}, \quad (51)
$$
$$
\frac{\mathbf{H}'_e}{c} = \gamma \left( \frac{\mathbf{H}_e}{c} + \frac{\mathbf{v} \times \mathbf{D}_e}{c} \right) - (\gamma - 1) \left( \hat{\mathbf{v}} \cdot \frac{\mathbf{H}_e}{c} \right) \hat{\mathbf{v}}. \quad (52)
$$
The densities of Gilbertian electric and magnetic dipole moments $P_m$ and $M_m$ comprise the tensor $P_m$,

$$
P_m = \begin{pmatrix}
0 & P_{m,x} & P_{m,y} & P_{m,z} \\
-P_{m,x} & 0 & M_{m,z}/c & -M_{m,y}/c \\
-P_{m,y} & -M_{m,z}/c & 0 & M_{m,x}/c \\
-P_{m,z} & M_{m,y}/c & -M_{m,x}/c & 0
\end{pmatrix}
$$

such that the fields in the $'$ frame where the frame of eq. (53) has velocity $v$ are

$$
P'_m = \gamma \left( P_m + \frac{v}{c} \times \frac{M_m}{c} \right) - (\gamma - 1)(\hat{v} \cdot P_m)\hat{v}, \quad (54)
$$

$$
\frac{M'_m}{c} = \gamma \left( \frac{M_m}{c} - \frac{v}{c} \times P_m \right) - (\gamma - 1) \left( \hat{v} \cdot \frac{M_m}{c} \right) \hat{v}, \quad (55)
$$

Finally, the macroscopic fields $D_m = E/\mu_0 - c^2 P_m$ and $H_m = B/\mu_0 + M_m$ can be written as components of the tensor $G_m$,

$$
G_m = \frac{F}{\mu_0} - c^2 P_m = \begin{pmatrix}
0 & D_{m,x} & D_{m,y} & D_{m,z} \\
-D_{m,x} & 0 & -cH_{m,z} & cH_{m,y} \\
-D_{m,y} & cH_{m,z} & 0 & -cH_{m,x} \\
-D_{m,z} & -cH_{m,y} & cH_{m,x} & 0
\end{pmatrix}, \quad (56)
$$

such that the fields in the $'$ frame where the frame of eq. (56) has velocity $v$ are

$$
D'_m = \gamma \left( D_m - \frac{v}{c} \times cH_m \right) - (\gamma - 1)(\hat{v} \cdot D_m)\hat{v}, \quad (57)
$$

$$
cH'_m = \gamma \left( cH_m + \frac{v}{c} \times D_m \right) - (\gamma - 1)(\hat{v} \cdot cH_m)\hat{v}. \quad (58)
$$

If we accept that the forces on charges $q$ and $p$ in their rest frame are

$$
F_e = \mu_0 qD_m, \quad F_m = \mu_0 pH_e, \quad (59)
$$
as in eq. (8), then we see by inverting eqs. (52) and (57) that the Lorentz forces in a frame where the charges have velocity $v$ are as in eqs. (17)-(18).

**C Appendix: Momentum Density and Stress Tensor**

We can extend an argument of Minkowski [41] as to field momentum by considering the total force density on electromagnetic media, following eqs. (17)-(18),

$$
f = f_e + f_m = \mu_0 (\rho_e D_m + \vec{J}_e \times H_m) + \mu_0 (\rho_m H_e - \vec{J}_m \times D_e) = \frac{dP_{\text{mech}}}{dt}, \quad (60)
$$
where $p_{\text{mech}}$ is the density of mechanical momentum in the media. Using the Maxwell equations (5) for the macroscopic fields,

$$\frac{dp_{\text{mech}}}{dt} = \mu_0 \left[ D_m(\nabla \cdot D_e) - H_m \times (\nabla \times H_e) + H_m \times \frac{\partial D_e}{\partial t} \right]$$

$$+ \mu_0 \left[ H_e(\nabla \cdot H_m) - D_e \times (\nabla \times D_m) - D_e \times \frac{\partial H_m}{\partial t} \right]$$

$$= -\frac{\partial}{\partial t} (\mu_0 D_e \times H_m) + \mu_0 [ D_m(\nabla \cdot D_e) + H_e(\nabla \cdot H_m)$$

$$- D_e \times (\nabla \times D_m) - H_m \times (\nabla \times H_e)]$$

$$\equiv -\frac{\partial p_{\text{EM}}}{\partial t} + \nabla \cdot T_{\text{EM}}, \quad (61)$$

where

$$p_{\text{EM}} = \mu_0 D_e \times H_m \quad (62)$$

is the density of momentum associated with the electromagnetic field, and for linear media,

$$T_{\text{EM},ij} = \mu_0 \left[ D_{m,i} D_{e,j} + H_{e,i} H_{m,j} - \delta_{ij} \frac{D_e \cdot D_m}{2} \right] \quad (63)$$

is the symmetric Maxwell stress 3-tensor associated with the electromagnetic fields. To arrive at eq. (63) we note that for linear media,

$$[D_m(\nabla \cdot D_e) - D_e \times (\nabla \times D_m)]_i = D_{m,i} \frac{\partial D_{e,j}}{\partial x_j} - D_{e,j} \frac{\partial D_{m,i}}{\partial x_j} + D_{e,i} \frac{\partial D_{m,j}}{\partial x_j}$$

$$= \frac{\partial}{\partial x_j} \left[ D_{m,i} D_{e,j} - \delta_{ij} \frac{D_e \cdot D_m}{2} \right]. \quad (64)$$

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**References**


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This paper mentions the earlier history of erratic results on this topic.


A sign error 6 lines before eq. (6) is corrected in http://arxiv.org/pdf/physics/0208007.pdf


